

COMPANION POINTS AND LOCALLY ANALYTIC SOCLE FOR $GL_2(L)$

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ABSTRACT. Let $p > 2$ be a prime number, and L be a finite extension of \mathbb{Q}_p , we prove Breuil's locally analytic socle conjecture for $GL_2(L)$, showing the existence of all the companion points on the definite (patched) eigenvariety. This work relies on infinitesimal “R=T” results for the patched eigenvariety and the comparison of (partially) de Rham families and (partially) Hodge-Tate families. This method allows in particular to find companion points of non-classical points.

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1. INTRODUCTION

In this paper, we prove (under mild technical hypotheses) Breuil's locally analytic socle conjecture for $\mathrm{GL}_2(L)$ where L is a finite extension of \mathbb{Q}_p .

Breuil's locally analytic socle conjecture for $\mathrm{GL}_2(L)$. Let L_0 be the maximal unramified extension over \mathbb{Q}_p in L , $d := [L : \mathbb{Q}_p]$, $d_0 := [L_0 : \mathbb{Q}_p]$, E be a finite extension of \mathbb{Q}_p big enough to contain all the \mathbb{Q}_p -embeddings of L in $\overline{\mathbb{Q}_p}$, and $\Sigma_L := \mathrm{Hom}_{\mathbb{Q}_p}(L, \overline{\mathbb{Q}_p}) = \mathrm{Hom}_{\mathbb{Q}_p}(L, E)$.

Let ρ_L be a two dimensional crystalline representation of Gal_L over E with distinct Hodge-Tate weights $(-k_{1,\sigma}, -k_{2,\sigma})_{\sigma \in \Sigma_L}$ ($k_{1,\sigma} > k_{2,\sigma}$) (where we use the convention that the Hodge-Tate weight of the cyclotomic character is -1), let $\alpha, \tilde{\alpha}$ be the eigenvalues of crystalline Frobenius φ^{d_0} on $D_{\mathrm{cris}}(\rho_L)$, and suppose $\alpha\tilde{\alpha}^{-1} \neq 1, p^{\pm d_0}$. Put $\delta := \mathrm{unr}(\alpha) \prod_{\sigma \in \Sigma_L} \sigma^{k_{1,\sigma}} \otimes \mathrm{unr}(\tilde{\alpha}) \prod_{\sigma \in \Sigma_L} \sigma^{k_{2,\sigma}}$ (as a character of $T(L) \cong L^\times \times L^\times$), and for $J \subseteq \Sigma_L$, put $\delta_J^c := \delta \left(\prod_{\sigma \in J} \sigma^{k_{2,\sigma} - k_{1,\sigma}} \otimes \prod_{\sigma \in J} \sigma^{k_{1,\sigma} - k_{2,\sigma}} \right)$ where $\mathrm{unr}(z)$ denotes the unramified character of L^\times sending uniformizers to z ; we define $\tilde{\delta}, \tilde{\delta}_J^c$ the same way as δ, δ_J^c by exchanging α and $\tilde{\alpha}$. Recall ρ_L is trianguline, and there exists $\Sigma \subseteq \Sigma_L$ (resp. $\tilde{\Sigma} \subseteq \Sigma_L$) such that δ_Σ^c is a trianguline parameter of ρ_L , also called a *refinement* of ρ_L . The refinement δ_Σ^c (resp. $\tilde{\delta}_\Sigma^c$) is called *non-critical* if $\Sigma = \emptyset$ (resp. $\tilde{\Sigma} = \emptyset$). Note the information of Σ and $\tilde{\Sigma}$ is lost when passing to the Weil-Deligne representation associated to ρ_L , thus is invisible in classical local Langlands correspondence. In fact, in terms of filtered φ -modules, we have

$$\begin{aligned} \Sigma &= \{ \sigma \in \Sigma_L \mid \mathrm{Fil}^i D_{\mathrm{dR}}(\rho_L)_\sigma \text{ is an eigenspace of } \varphi^{d_0} \text{ of eigenvalue } \alpha, -k_{1,\sigma} < i \leq -k_{2,\sigma} \}, \\ \tilde{\Sigma} &= \{ \sigma \in \Sigma_L \mid \mathrm{Fil}^i D_{\mathrm{dR}}(\rho_L)_\sigma \text{ is an eigenspace of } \varphi^{d_0} \text{ of eigenvalue } \tilde{\alpha}, -k_{1,\sigma} < i \leq -k_{2,\sigma} \}, \end{aligned}$$

where $D_{\mathrm{dR}}(\rho_L) \cong D_{\mathrm{cris}}(\rho_L) \otimes_{L_0} L \cong \otimes_{\sigma \in \Sigma_L} D_{\mathrm{dR}}(\rho_L)_\sigma$ is naturally equipped with an E -linear action of φ^{d_0} , and $\mathrm{Fil}^i D_{\mathrm{dR}}(\rho_L)_\sigma$ is the one dimensional non-trivial Hodge filtration of $D_{\mathrm{dR}}(\rho_L)_\sigma$ for $-k_{1,\sigma} < i \leq -k_{2,\sigma}$.

For a continuous very regular character χ (cf. (3)) of $T(L)$ over E , put $I(\chi) := \mathrm{soc}(\mathrm{Ind}_{\overline{B}(L)}^{\mathrm{GL}_2(L)} \chi)^{\mathbb{Q}_p\text{-an}}$, which is an irreducible locally \mathbb{Q}_p -analytic representation of $\mathrm{GL}_2(L)$ over E . Note if the weight of χ is dominant, then $I(\chi)$ is locally algebraic. Put $\chi^\sharp := \chi(\mathrm{unr}(p^{-d_0}) \otimes \prod_{\sigma \in \Sigma_L} \sigma)$.

Suppose ρ_L is the restriction of certain global modular Galois representation ρ (i.e. ρ is associated to classical automorphic representations; in this paper, we would consider the case of automorphic representations of definite unitary groups). Using global method, to ρ , could be associated (at least in the case we consider) an admissible unitary Banach representation $\hat{\Pi}(\rho)$ of $\mathrm{GL}_2(L)$ with $I(\delta^\sharp \delta_B^{-1}) \cong I(\tilde{\delta}^\sharp \tilde{\delta}_B^{-1}) \hookrightarrow \hat{\Pi}(\rho)$ where $\delta_B = \mathrm{unr}(p^{-d_0}) \otimes \mathrm{unr}(p^{d_0})$ is the modulus character of the Borel subgroup B (of upper triangular matrices). The representation $\hat{\Pi}(\rho)$ is an important object in p -adic Langlands program (cf. [9]). We have the following Breuil's conjecture (cf. [8, Conj.8.1] and [10]) concerning the socle of the locally \mathbb{Q}_p -analytic representation $\hat{\Pi}(\rho)^{\mathrm{an}}$ (which is dense inside $\hat{\Pi}(\rho)$).

Conjecture 1.1 (Breuil). *For $\chi : T(L) \rightarrow E^\times$, $I(\chi) \hookrightarrow \hat{\Pi}(\rho)^{\mathrm{an}}$ if and only if $\chi = (\delta_J^c)^\sharp \delta_B^{-1}$ for $J \subseteq \Sigma$ or $(\tilde{\delta}_J^c)^\sharp \tilde{\delta}_B^{-1}$ for $J \subseteq \tilde{\Sigma}$.*

The “only if” part would be (in many cases) a consequence of global triangulation theory, while the “if” part is more difficult. In modular curve case (thus $L = \mathbb{Q}_p$), this was proved in [7] using p -adic comparison theorems and the theory of overconvergent modular forms. In [3], Bergdall reproved the result of Breuil-Emerton by studying the geometry of Coleman-Mazur eigencurve at classical points. In the author's thesis ([18]), some partial results (especially for $|J| = 1$) were obtained in unitary Shimura curves case, by showing the existence of overconvergent companion forms over unitary Shimura curves following the strategy of Breuil-Emerton. In this paper, we completely prove this conjecture in definite

unitary groups case (with $p > 2$), by studying the geometry of (certain stratifications of) the patched eigenvariety of Breuil-Hellmann-Schraen ([12]) at *possibly-non-classical* points.

Main results. Let F^+ be a totally real field, and suppose for simplicity in the introduction that there's only one place u of F^+ above p ; let F be a quadratic imaginary unramified extension of F^+ such that u is split in F , fix a place \tilde{u} of F above F^+ , and to be consistent with the notation in the precedent section, let L be $F_{\tilde{u}} \cong F_u^+$. Let G be a two variables quasi-split definite unitary group associated to F/F^+ with $G(F_u^+) \cong \mathrm{GL}_2(F_u^+) \cong \mathrm{GL}_2(L)$. We fix a compact open subgroup $U^p = \prod_{v \nmid p, \infty} U_v$ of $G(\mathbb{A}_{F^+}^{\infty, u})$ with U_v a compact open subgroup of $G(F_v^+)$, and put

$$\widehat{S}(U^p, E) := \{f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty})/U^p \rightarrow E \mid f \text{ is continuous}\}$$

which is an E -Banach space equipped with a continuous unitary action of $\mathrm{GL}_2(L)$ and of some commutative Hecke algebra \mathcal{H}^p over \mathcal{O}_E outside p (where \mathcal{H}^p would be big enough to determine Galois representations).

Let ρ be a two dimensional continuous representation of Gal_F over E associated to classical automorphic representations of G and suppose to ρ is associated a maximal ideal \mathfrak{m}_ρ of $\mathcal{H}^p \otimes_{\mathcal{O}_E} E$; suppose moreover $\overline{\rho}$ is absolutely irreducible and $\overline{\rho}|_{\mathrm{Gal}_F(\zeta_p)}$ is adequate ([32]) with $\overline{\rho}$ (the semi-simplification of) the reduction of ρ over k_E (the residue field of E). Suppose $\rho_L := \rho|_{\mathrm{Gal}_L}$ is crystalline satisfying the condition in the precedent section, and we use the notation of the precedent section. The main result is:

Theorem 1.2. *Suppose $p > 2$, Conj.1.1 is true for $\widehat{\Pi}(\rho) := \widehat{S}(U^p, E)[\mathfrak{m}_\rho]$ (the maximal subspace of $\widehat{S}(U^p, E)$ killed by \mathfrak{m}_ρ).*

By Emerton's method ([21]), one can construct an eigenvariety $\mathcal{E}(U^p)$ from $\widehat{S}(U^p, E)$. The closed points of $\mathcal{E}(U^p)$ can be parameterized by (ρ', δ') where ρ' is a semi-simple continuous representation of Gal_F over \overline{E} to which could be associated a maximal ideal $\mathfrak{m}_{\rho'}$ of \mathcal{H}^p , and where δ' is a continuous character of $T(L)$ over \overline{E} ; moreover, $(\rho', \delta') \in \mathcal{E}(U^p)(\overline{E})$ if and only if the corresponding eigenspace

$$J_B(\widehat{S}(U^p, E)^{\mathrm{an}})[\mathfrak{m}_{\rho'}, T(L) = \delta'] \neq 0,$$

where $J_B(\cdot)$ denotes the Jacquet-Emerton functor for locally analytic representations. By the very construction and adjunction property of $J_B(\cdot)$, one can deduce from Thm.1.2:

Corollary 1.3. *Let χ be a continuous character of $T(L)$ in E^\times , $(\rho, \chi) \in \mathcal{E}(U^p)(\overline{E})$ if and only if $\chi = (\delta_J^c)^\sharp$ for $J \subseteq \Sigma$ or $\chi = (\tilde{\delta}_J^c)^\sharp$ for $J \subseteq \tilde{\Sigma}$.*

In fact, we also get a similar result in trianguline case (cf. Cor.4.7). The points $(\rho, (\delta_J^c)^\sharp)$ for $J \subseteq \Sigma$ (resp. $(\rho, (\tilde{\delta}_J^c)^\sharp)$ for $J \subseteq \tilde{\Sigma}$) are called *companion points* of the classical point (ρ, δ^\sharp) (resp. of $(\rho, \tilde{\delta}^\sharp)$). We refer to §4.1 for more discussion on the relation between Breuil's locally analytic socle conjecture and the existence of companion points.

Strategy of the proof. Suppose $p > 2$, by [14], using Taylor-Wiles-Kisin patching method, one obtains an R_∞ -admissible unitary Banach representation Π_∞ of $\mathrm{GL}_2(L)$, where R_∞ is the usual patched deformation ring of $\overline{\rho}$; a point is that one can recover $\widehat{S}(U^p, E)_{\overline{\rho}}$ (the localisation at $\overline{\rho}$) from Π_∞ , in particular, to prove Conj.1.1, it's sufficient to prove a similar result for $\Pi_\infty[\mathfrak{m}_\rho]$ (where we also use \mathfrak{m}_ρ to denote the maximal ideal of $R_\infty[1/p]$ corresponding to ρ).

The patched eigenvariety $X_p(\overline{\rho})$ (cf. [12]) is a rigid space over E with points parameterized by $(\mathfrak{m}_y, \delta')$ where \mathfrak{m}_y is a maximal ideal of $R_\infty[1/p]$, and δ' is a character of $T(L)$, moreover $(\mathfrak{m}_y, \delta') \in X_p(\overline{\rho})$ if and only if the eigenspace

$$J_B(\Pi_\infty^{R_\infty - \mathrm{an}})[\mathfrak{m}_y, T(L) = \delta'] \neq 0,$$

where “ $R_\infty\text{-an}$ ” denotes the R_∞ -analytic vectors defined in [12, §3.1]. For $J \subseteq \Sigma_L$, put $\underline{\Delta}_J := (k_{1,\sigma}, k_{2,\sigma} + 1)_{\sigma \in J}$, and $L(\underline{\Delta}_J)$ to be the irreducible algebraic representation of $\text{Res}_{L/\mathbb{Q}_p} \text{GL}_2$ with highest weight $\underline{\Delta}_J$, which is thus a locally J -analytic representation of $\text{GL}_2(L)$. Put

$$\Pi_\infty^{R_\infty\text{-an}}(\underline{\Delta}_J) := (\Pi_\infty^{R_\infty\text{-an}} \otimes_E L(\underline{\Delta}_J)')^{\Sigma_L \setminus J\text{-an}} \otimes L(\underline{\Delta}_J),$$

which is a closed subrepresentation of Π_∞ , and from which we can construct a rigid closed subspace $X_p(\overline{\rho}, \underline{\Delta}_J)$ of $X_p(\overline{\rho})$ such that $(y, \delta') \in X_p(\overline{\rho}, \underline{\Delta}_J)$ if and only if the eigenspace

$$J_B(\Pi_\infty^{R_\infty\text{-an}}(\underline{\Delta}_J))[\mathfrak{m}_y, T(L) = \delta'] \neq 0.$$

Denote by \widehat{T}_L the rigid space over E parameterizing continuous characters of $T(L)$, and put $\widehat{T}_L(\underline{\Delta}_J)$ to be the closed subspace of characters χ with $\text{wt}(\chi)_\sigma = (k_{1,\sigma}, k_{2,\sigma} + 1)$ for $\sigma \in J$. By construction, we have the following commutative diagram (which is *not* Cartesian in general)

$$\begin{array}{ccc} X_p(\overline{\rho}, \underline{\Delta}_J) & \longrightarrow & X_p(\overline{\rho}) \\ \downarrow & & \downarrow \\ \widehat{T}_L(\underline{\Delta}_J) & \longrightarrow & \widehat{T}_L \end{array}$$

where the vertical maps send (y, δ') to δ' . For $J' \subseteq J$, $\Pi_\infty^{R_\infty\text{-an}}(\underline{\Delta}_{J'})$ is a subrepresentation of $\Pi_\infty^{R_\infty\text{-an}}(\underline{\Delta}_J)$, and hence $X_p(\overline{\rho}, \underline{\Delta}_{J'})$ is a rigid closed subspace of $X_p(\overline{\rho}, \underline{\Delta}_J)$ and the following diagram commutes

$$\begin{array}{ccc} X_p(\overline{\rho}, \underline{\Delta}_J) & \longrightarrow & X_p(\overline{\rho}, \underline{\Delta}_{J'}) \\ \downarrow & & \downarrow \\ \widehat{T}_L(\underline{\Delta}_J) & \longrightarrow & \widehat{T}_L(\underline{\Delta}_{J'}); \end{array}$$

put $X_p(\overline{\rho}, \underline{\Delta}_J, J') := X_p(\overline{\rho}, \underline{\Delta}_{J'}) \times_{\widehat{T}_L(\underline{\Delta}_{J'})} \widehat{T}_L(\underline{\Delta}_J)$, thus $X_p(\overline{\rho}, \underline{\Delta}_J)$ is a closed subspace of $X_p(\overline{\rho}, \underline{\Delta}_J, J')$.

Now let $J \subseteq \Sigma$ (the case for $\widetilde{\Sigma}$ is the same), and suppose there exists an injection $I((\delta_J^c)^\# \delta_B^{-1}) \hookrightarrow \Pi_\infty^{R_\infty\text{-an}}[\mathfrak{m}_\rho]$ (which automatically factors through $\Pi_\infty^{R_\infty\text{-an}}(\underline{\Delta}_{\Sigma_L \setminus J})[\mathfrak{m}_\rho]$); suppose $J \neq \Sigma$, we would prove there exists an injection $I((\delta_{J \cup \{\sigma\}}^c)^\# \delta_B^{-1}) \hookrightarrow \Pi_\infty^{R_\infty\text{-an}}[\mathfrak{m}_\rho]$ for all $\sigma \in \Sigma \setminus J$, from which (the “if” part of) Conj.1.1 follows (as mentioned before, the “only if” part is an easy consequence of the global triangulation theory) by induction on J (note the $(J = \emptyset)$ -case is known *a priori* by classical local Langlands correspondence). By representation theory, for $S \subseteq \Sigma_L$, we have in fact

- $I((\delta_S^c)^\# \delta_B^{-1}) \hookrightarrow \Pi_\infty^{R_\infty\text{-an}}[\mathfrak{m}_\rho]$ if and only $x_S^c := (\mathfrak{m}_\rho, (\delta_S^c)^\#) \in X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus S})$.

So by assumption, we have a point $x_J^c \in X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus J})$, and it's sufficient to find the point $x_{J \cup \{\sigma\}}^c$ inside $X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus (J \cup \{\sigma\})})$ for $\sigma \in \Sigma \setminus J$:

Theorem 1.4. *Keep the above notation.*

(1) *The rigid space $X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus J})$ is smooth at the point x_J^c .*

(2) *The following statements are equivalent:*

- (a) $\sigma \in \Sigma \setminus J$;
- (b) *the natural projection of complete noetherian local E -algebras*

$$\widehat{\mathcal{O}}_{X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus J}, \Sigma_L \setminus (J \cup \{\sigma\})), x_J^c} \twoheadrightarrow \widehat{\mathcal{O}}_{X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus J}), x_J^c}$$

induced by the closed embedding $X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus J}) \hookrightarrow X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus J}, \Sigma_L \setminus (J \cup \{\sigma\}))$ is not an isomorphism;

- (c) $x_{J \cup \{\sigma\}}^c \in X_p(\overline{\rho}, \underline{\Delta}_{\Sigma_L \setminus (J \cup \{\sigma\})})$.

The smoothness of $X_p(\bar{\rho}, \Delta_{\Sigma_L \setminus J})$ follows from the same arguments of [13] (see also [5]). A key point is obtaining (a bound for) the dimension of the tangent space of $X_p(\bar{\rho}, \Delta_{\Sigma_L \setminus J})$ at x_J^c via Galois cohomology calculation (in our case, it would in particular involve some partially de Rham Galois cohomology considered in [19]).

For (2) (from which Thm.1.2 follows), the direction (c) \Rightarrow (a) follows easily from global triangulation theory; (b) \Rightarrow (c) follows from some locally analytic representation theory (e.g. Breuil's adjunction formula) and some commutative algebra arguments (cf. Thm.4.4). The equivalence between (a) and (b) is rather a consequence of infinitesimal “ $R = T$ ” results, which we explain in more details for the rest of the introduction:

Denote by R_∞^p the “prime-to- p ” part of R_∞ (where $R_\infty \cong R_\infty^p \times R_{\bar{\rho}_p}^\square$); by [12, Thm.1.1], we have a natural embedding

$$(1) \quad X_p(\bar{\rho}) \hookrightarrow (\mathrm{Spf} R_\infty^p)^{\mathrm{rig}} \times X_{\mathrm{tri}}^\square(\bar{\rho}_p)$$

which is moreover a local isomorphism, where $\bar{\rho}_p := \bar{\rho}|_{\mathrm{Gal}_L}$ and $X_{\mathrm{tri}}^\square(\bar{\rho}_p)$ is the trianguline variety (cf. [24], [12, §2]) whose closed points can be parameterized as (ρ'_p, δ') where ρ'_p is a framed deformation of $\bar{\rho}_p$ over \bar{E} , δ' is a continuous character of $T(L)$. Using partially de Rham data, one can get a stratification of $X_{\mathrm{tri}}^\square(\bar{\rho})$: for $S' \subseteq S \subseteq \Sigma_L$, one can get a closed rigid subspace $X_{\mathrm{tri}, S' - \mathrm{dR}}^\square(\bar{\rho}, \Delta_S)$ of $X_{\mathrm{tri}}^\square(\bar{\rho})$ satisfying in particular $(\rho'_p, \delta') \in X_{\mathrm{tri}, S' - \mathrm{dR}}^\square(\bar{\rho}, \Delta_S)$ if and only if ρ'_p is S' -de Rham and $\mathrm{wt}(\delta')_\sigma = (k_{1,\sigma}, k_{2,\sigma})$ for $\sigma \in S$ (in particular, ρ'_p is S' -de Rham and $S \setminus S'$ -Hodge-Tate). For $\sigma \in \Sigma_L \setminus J$, we can prove (cf. Thm.3.22) that the local isomorphism (1) induces a commutative diagram

$$\begin{array}{ccc} X_p(\bar{\rho}, \Delta_{\Sigma_L \setminus J}) & \longrightarrow & (\mathrm{Spf} R_\infty^p)^{\mathrm{rig}} \times X_{\mathrm{tri}, \Sigma_L \setminus J - \mathrm{dR}}^\square(\bar{\rho}, \Delta_{\Sigma_L \setminus J}) \\ \downarrow & & \downarrow \\ X_p(\bar{\rho}, \Delta_{\Sigma_L \setminus J}, (J \cup \{\sigma\})) & \longrightarrow & (\mathrm{Spf} R_\infty^p)^{\mathrm{rig}} \times X_{\mathrm{tri}, \Sigma_L \setminus (J \cup \{\sigma\}) - \mathrm{dR}}^\square(\bar{\rho}, \Delta_{\Sigma_L \setminus J}) \end{array},$$

and moreover the horizontal maps are local isomorphisms at x_J^c (note also the vertical maps are closed embeddings). Denote by z_J^c the image of x_J^c in $X_{\mathrm{tri}}^\square(\bar{\rho}_p)$, the equivalence of (a) and (b) thus follows from (cf. Cor.2.6, Cor.3.24):

- $\sigma \in \Sigma \setminus J$ if and only if the complete local rings of $X_{\mathrm{tri}, \Sigma_L \setminus J - \mathrm{dR}}^\square(\bar{\rho}, \Delta_{\Sigma_L \setminus J})$ (which is a $\Sigma_L \setminus J$ -de Rham family) and $X_{\mathrm{tri}, \Sigma_L \setminus (J \cup \{\sigma\}) - \mathrm{dR}}^\square(\bar{\rho}, \Delta_{\Sigma_L \setminus J})$ (which is a $\Sigma_L \setminus (J \cup \{\sigma\})$ -de Rham and σ -Hodge-Tate family) at z_J^c are different,

which is a pure Galois result and follows by Galois cohomology calculations.

Let's remark the particular global context that we are working in is not important for the above arguments. For example, assuming similar patching result for the completed H^1 of Shimura curves, one can probably prove Breuil's locally analytic socle conjecture in that case using the same arguments. On the other hand, by comparison of eigenvarieties, it might be also possible to deduce from Cor.1.3 the existence of overconvergent Hilbert companion forms. We refer to the body of the text for more detailed and more precise statements (with slightly different notations).

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2. TRIANGULINE VARIETY REVISITED

2.1. Trianguline variety and some stratifications. Let L be a finite extension of \mathbb{Q}_p , \mathcal{O}_L the ring of integers of L , ϖ_L a uniformizer of \mathcal{O}_L , $d_L := [L : \mathbb{Q}_p]$, L_0 the maximal unramified extension over \mathbb{Q}_p in L , $q_L := |\mathcal{O}_L/\varpi_L| = p^{[L_0:\mathbb{Q}_p]}$, Gal_L the absolute Galois group of L , and Σ_L the set of \mathbb{Q}_p -embeddings of L in $\overline{\mathbb{Q}_p}$.

Let E be a finite extension of \mathbb{Q}_p big enough containing all the embeddings of L in $\overline{\mathbb{Q}_p}$, \mathcal{O}_E be the ring of integers of E , ϖ_E a uniformizer of \mathcal{O}_E , $k_E := \mathcal{O}_E/\varpi_E$. Let $\bar{\tau}_L := \text{Gal}_L \rightarrow \text{GL}_2(k_E)$ be a two dimensional continuous representation of Gal_L over k_E . Denote by $R_{\bar{\tau}_L}^\square$ the framed deformation ring, which is a complete noetherian local \mathcal{O}_E -algebra, and put $X_{\bar{\tau}_L}^\square := (\text{Spf } R_{\bar{\tau}_L}^\square)^{\text{rig}}$, which is thus a rigid space over E .

Denote by \hat{T}_L the rigid space over E parameterizing continuous characters of $T(L)$, in particular

$$\hat{T}_L(\overline{E}) = \{\delta : T(L) \rightarrow \overline{E}^\times, \delta \text{ is continuous}\}.$$

A continuous character $\delta = \delta_1 \otimes \delta_2 : T(L) \rightarrow \overline{E}^\times$ is called *regular* if (where $\text{unr}(z)$ denotes the unramified character of L^\times sending uniformizers to z)

$$(2) \quad \begin{cases} \delta_i \delta_j^{-1} \neq \prod_{\sigma \in \Sigma_L} \sigma(z)^{k_\sigma} \text{ for all } \underline{k}_{\Sigma_L} = (k_\sigma)_{\sigma \in \Sigma_L} \in \mathbb{Z}_{\geq 0}^{d_L}, i \neq j, \\ \delta_i \delta_j^{-1} \neq \text{unr}(q_L) \prod_{\sigma \in \Sigma_L} \sigma(z)^{k_\sigma} \text{ for } \underline{k}_{\Sigma_L} \in \mathbb{Z}_{\geq 1}^{d_L}, i \neq j; \end{cases}$$

δ is called *very regular* if

$$(3) \quad \begin{cases} \delta_1 \delta_2^{-1} \neq \prod_{\sigma \in \Sigma_L} \sigma(z)^{k_\sigma} \text{ for all } \underline{k}_{\Sigma_L} = (k_\sigma)_{\sigma \in \Sigma_L} \in \mathbb{Z}^{d_L}, \\ \delta_i \delta_j^{-1} \neq \text{unr}(q_L) \prod_{\sigma \in \Sigma_L} \sigma(z)^{k_\sigma} \text{ for } \underline{k}_{\Sigma_L} \in \mathbb{Z}^{d_L}, i \neq j. \end{cases}$$

Let $\text{wt}(\delta) = (\text{wt}(\delta_1)_\sigma, \text{wt}(\delta_2)_\sigma)_{\sigma \in \Sigma_L} \in \overline{E}^{2|d_L|}$ be the weight of δ , $\text{wt}(\delta)$ is called *dominant* (resp. *strictly dominant*) if $\text{wt}(\delta_1)_\sigma - \text{wt}(\delta_2)_\sigma \in \mathbb{Z}_{\geq 0}$ (resp. $\text{wt}(\delta_1)_\sigma - \text{wt}(\delta_2)_\sigma \in \mathbb{Z}_{\geq 1}$) for all $\sigma \in \Sigma_L$. Thus if $\text{wt}(\delta)$ is strictly dominant, then δ is regular if and only if δ is very regular.

Let \hat{T}_L^{reg} be the subset of $\hat{T}_L(\overline{E})$ of regular characters, which is in fact Zariski-open in \hat{T}_L , and put

$$U_{\text{tri}}^{\square, \text{reg}} := \{(r, \delta) \in X_{\bar{\tau}_L}^\square(\overline{E}) \times \hat{T}_L^{\text{reg}} \mid r \text{ is trianguline of parameter } \delta\}.$$

Following [12, Def.2.4], let $X_{\text{tri}}^\square(\bar{\tau}_L) \hookrightarrow X_{\bar{\tau}_L}^\square \times \hat{T}_L$ be the (reduced) Zariski-closure of $U_{\text{tri}}^{\square, \text{reg}}$ in $X_{\bar{\tau}_L}^\square \times \hat{T}_L$. Recall

Theorem 2.1 ([12, Thm.2.6 (i)]). $X_{\text{tri}}^\square(\bar{\tau}_L)$ is equidimensional of dimension $4 + 3d_L$.

Let A be an artinian local E -algebra, r_A a free A -module of rank n equipped with a continuous action of Gal_L . Using the isomorphism

$$L \otimes_{\mathbb{Q}_p} A \xrightarrow{\sim} \bigoplus_{\sigma \in \Sigma_L} A, \quad a \otimes b \mapsto (\sigma(a)b)_{\sigma \in \Sigma_L},$$

$D_{\text{dR}}(r_A) := (B_{\text{dR}} \otimes_{\mathbb{Q}_p} r_A)^{\text{Gal}_L}$ admits a decomposition $D_{\text{dR}}(r_A) \xrightarrow{\sim} \bigoplus_{\sigma \in \Sigma_L} D_{\text{dR}}(r_A)_\sigma$. For $\sigma \in \Sigma_L$, r_A is called σ -de Rham if $D_{\text{dR}}(r_A)_\sigma$ is a free A -module of rank n ; for $J \subseteq \Sigma_L$, r_A is called J -de Rham if r_A is σ -de Rham for all $\sigma \in J$; r_A is thus de Rham if r_A is Σ_L -de Rham.

Let $J \subseteq \Sigma_L$, $\underline{k}_J := (k_{1,\sigma}, k_{2,\sigma})_{\sigma \in J} \in \mathbb{Z}^{2|J|}$, and suppose $k_{1,\sigma} > k_{2,\sigma}$ for all $\sigma \in J$. Denote by $\hat{T}_L(\underline{k}_J)$ the reduced closed subspace of \hat{T}_L with (where $\text{wt}(\delta_i) = (\text{wt}(\delta_i)_\sigma)_{\sigma \in \Sigma_L} \in \overline{E}^{d_L}$ denotes the weight of δ_i)

$$\hat{T}_L(\underline{k}_J)(\overline{E}) = \{\delta = \delta_1 \otimes \delta_2 \in \hat{T}_L(\overline{E}) \mid \text{wt}(\delta_i)_\sigma = k_{i,\sigma}, \forall \sigma \in J, i = 1, 2\}.$$

Let $X_{\text{tri}}^\square(\bar{\tau}_L, \underline{k}_J) := X_{\text{tri}}^\square(\bar{\tau}_L) \times_{\hat{T}_L} \hat{T}_L(\underline{k}_J)$, note that for $(r, \delta) \in X_{\text{tri}}^\square(\bar{\tau}_L, \underline{k}_J)(\overline{E})$, the Hodge-Tate weights of r at $\sigma \in J$ is $(-k_{1,\sigma}, -k_{2,\sigma})$ (e.g. see [12, Prop.2.9], where we use the convention that the Hodge-Tate

weight of the cyclotomic character is -1). Let $X_{\text{tri}, J\text{-dR}}^\square(\bar{r}_L, \underline{k}_J)'$ denote the closed subspace of $X_{\text{tri}}^\square(\bar{r}_L)$ satisfying that

- for any artinian local E -algebra A , $\text{Spec } A \rightarrow X_{\text{tri}}^\square(\bar{r}_L)$, the associated Gal_L -representation r_A is J -de Rham of Hodge-Tate weights $(-k_{1,\sigma}, -k_{2,\sigma})_{\sigma \in J}$ (i.e. $\text{Fil}^{-k_{i,\sigma}} D_{\text{dR}}(r_A)_\sigma / \text{Fil}^{-k_{i,\sigma}+1} D_{\text{dR}}(r_A)_\sigma$ is a free A -module of rank 1 for all $i = 1, 2, \sigma \in J$),

whose existence is ensured by [31, Thm.2]. Let

$$(4) \quad X_{\text{tri}, J\text{-dR}}^\square(\bar{r}_L, \underline{k}_J) := X_{\text{tri}, J\text{-dR}}^\square(\bar{r}_L, \underline{k}_J)' \times_{X_{\text{tri}}^\square(\bar{r}_L)} X_{\text{tri}}^\square(\bar{r}_L, \underline{k}_J) \cong X_{\text{tri}, J\text{-dR}}^\square(\bar{r}_L, \underline{k}_J) \times_{\hat{T}_L} \hat{T}_L(\underline{k}_J).$$

which is a closed subspace of $X_{\text{tri}}^\square(\bar{r}_L, \underline{k}_J)$ and of $X_{\text{tri}, J\text{-dR}}^\square(\bar{r}_L, \underline{k}_J)'$. For $J' \subset J$, we have $X_{\text{tri}}^\square(\bar{r}_L, \underline{k}_J) \cong X_{\text{tri}}^\square(\bar{r}_L, \underline{k}_{J'}) \times_{\hat{T}_L(\underline{k}_{J'})} \hat{T}_L(\underline{k}_J)$, and we put

$$(5) \quad X_{\text{tri}, J'\text{-dR}}^\square(\bar{r}_L, \underline{k}_J) := X_{\text{tri}, J'\text{-dR}}^\square(\bar{r}_L, \underline{k}_{J'}) \times_{\hat{T}_L(\underline{k}_{J'})} \hat{T}_L(\underline{k}_J).$$

Thus for any finite extension E' of E , we have

$$X_{\text{tri}, J'\text{-dR}}^\square(\bar{r}_L, \underline{k}_J)(E') = \{(r, \delta_1 \otimes \delta_2) \in X_{\text{tri}}^\square(\bar{r}_L)(E') \mid \forall \sigma \in J, \text{wt}(\delta_i)_\sigma = k_{i,\sigma}, \text{ and } r \text{ is } J'\text{-de Rham}\}.$$

By definitions, we have the following commutative diagram (compare with (19) below)

$$(6) \quad \begin{array}{ccccccccc} X_{\text{tri}, J\text{-dR}}^\square(\bar{r}_L, \underline{k}_J) & \longrightarrow & X_{\text{tri}, J'\text{-dR}}^\square(\bar{r}_L, \underline{k}_J) & \longrightarrow & X_{\text{tri}, J'\text{-dR}}^\square(\bar{r}_L, \underline{k}_{J'}) & \longrightarrow & X_{\text{tri}}^\square(\bar{r}_L, \underline{k}_{J'}) & \longrightarrow & X_{\text{tri}}^\square(\bar{r}_L) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{T}_L(\underline{k}_J) & \longrightarrow & \hat{T}_L(\underline{k}_J) & \longrightarrow & \hat{T}_L(\underline{k}_{J'}) & \longrightarrow & \hat{T}_L(\underline{k}_{J'}) & \longrightarrow & \hat{T}_L \end{array}$$

where the horizontal maps are all closed embeddings, and the second and fourth square is cartesian. For a closed subspace X of $X_{\text{tri}}^\square(\bar{r}_L)$, put $X(\underline{k}_J) := X \times_{X_{\text{tri}}^\square(\bar{r}_L)} X_{\text{tri}}^\square(\bar{r}_L, \underline{k}_J) \cong X \times_{\hat{T}_L} \hat{T}_L(\underline{k}_J)$, $X_{J'\text{-dR}}(\underline{k}_J) := X \times_{X_{\text{tri}}^\square(\bar{r}_L)} X_{\text{tri}, J'\text{-dR}}^\square(\bar{r}_L, \underline{k}_J)$. At last, we end this section by recalling the following definition.

Definition 2.2 ([13, Def.2.12]). *Let X be a union of irreducible components of an open subset of $X_{\text{tri}}^\square(\bar{r}_L)$ (over E), $x = (r, \delta)$ a closed point in $X_{\text{tri}}^\square(\bar{r}_L)$. Then X satisfies the accumulation property at x if x lies in X and if for any positive real number $C > 0$, the set of closed points $x' = (r', \delta' = \delta'_1 \otimes \delta'_2)$ satisfying*

- r' is crystalline, and the eigenvalues of the crystalline Frobenius $\varphi^{[L_0:\mathbb{Q}_p]}$ on $D_{\text{cris}}(r')$ are distinct;
- δ' is strictly dominant;
- r' is trianguline of parameter δ'
- $\text{wt}(\delta'_1)_\sigma - \text{wt}(\delta'_2)_\sigma > C$ for all $\sigma \in \Sigma_L$;

accumulates at x in X in the sense of [2, §3.3.1].

2.2. Tangent spaces. For a rigid space X over E , x a closed point in X , denote by $k(x)$ the residue field at x , and $T_{X,x}$ the tangent space of X at x , with can be identified with the $k(x)$ -vector space of morphisms $\text{Spec } k(x)[\epsilon]/\epsilon^2 \rightarrow X$ with the induced map $\text{Spec } k(x) \rightarrow X$ corresponding to x . For a locally algebraic character $\delta = \delta_1 \otimes \delta_2$ of $T(L)$ in \bar{E}^\times , put $N(\delta) := \{\sigma \in \Sigma_L \mid \text{wt}(\delta_1)_\sigma \geq \text{wt}(\delta_2)_\sigma\}$.

Let $x = (r, \delta = \delta_1 \otimes \delta_2)$ be a closed point in $X_{\text{tri}}^\square(\bar{r}_L)$. Suppose δ is locally algebraic and very regular with $\text{wt}(\delta_1)_\sigma \neq \text{wt}(\delta_2)_\sigma$ for all $\sigma \in \Sigma_L$. Let $J \subseteq N(\delta)$, $\underline{k}_J := (k_{1,\sigma}, k_{2,\sigma})_{\sigma \in J}$ with $k_{i,\sigma} = \text{wt}(\delta_i)_\sigma$. Suppose moreover r is J -de Rham (of Hodge-Tate weights $(-k_{1,\sigma}, -k_{2,\sigma})$ for $\sigma \in J$). Recall there exists $\Sigma(x) \subseteq N(\delta)$ such that r admits a triangulation of parameter

$$\delta' = \delta'_1 \otimes \delta'_2 = \delta \prod_{\sigma \in \Sigma(x)} (\sigma^{\text{wt}(\delta_2)_\sigma - \text{wt}(\delta_1)_\sigma} \otimes \sigma^{\text{wt}(\delta_1)_\sigma - \text{wt}(\delta_2)_\sigma});$$

roughly speaking, the set $\Sigma(x)$ measures the criticalness of the point x . Note also that $\Sigma(x) = N(\delta) \setminus N(\delta')$. Put $N(x) := N(\delta') \subseteq N(\delta)$, $C(x) := \Sigma_L \setminus N(x)$.

By assumption, x is a closed point of $X_{\text{tri}, J-\text{dR}}^\square(\bar{\tau}_L, \underline{k}_J) \hookrightarrow X_{\text{tri}}^\square(\bar{\tau}_L, \underline{k}_J) \hookrightarrow X_{\text{tri}}^\square(\bar{\tau}_L)$. Let X be a union of irreducible components of an open subset of $X_{\text{tri}}^\square(\bar{\tau}_L)$ such that X satisfies the accumulation property at x . The following theorem is due to Breuil-Hellmann-Schraen (cf. [13, §4]).

Theorem 2.3. *Keep the above notation, then*

- (1) $\dim_{k(x)} T_{X,x} = 4 + 3d_L$;
- (2) $\dim_{k(x)} T_{X(\underline{k}_J),x} = 4 + 3d_L - 2|J \cap (\Sigma_L \setminus \Sigma(x))| - |J \cap \Sigma(x)|$;

Which together with Thm.2.1 implies:

Corollary 2.4. *The rigid space X is smooth at the point x .*

We would give the proof of Thm.2.3 for the convenience of the reader and the author. Indeed, from this proof together with some results in [19], we would also get:

Theorem 2.5. *Keep the above notation, let $J' \subset J$, then*

- (1) $\dim_{k(x)} T_{X_{J-\text{dR}}(\underline{k}_J),x} = 4 + 3d_L - 2|J|$;
- (2) $\dim_{k(x)} T_{X_{J'-\text{dR}}(\underline{k}_J),x} = 4 + 3d_L - 2|J'| - 2|(J \setminus J') \cap (\Sigma_L \setminus \Sigma(x))| - |(J \setminus J') \cap \Sigma(x)|$.

Corollary 2.6. *Keep the notation in Thm.2.5, if $(J \cap J') \cap \Sigma(x) \neq \emptyset$, then $X_{J-\text{dR}}(\underline{k}_J)$ is a proper closed subspace of $X_{J'-\text{dR}}(\underline{k}_J)$.*

The rest of the section is devoted to the proof of Thm.2.3 and 2.5.

Since $X_{\text{tri}}^\square(\bar{\tau}_L)$ is equidimensional of dimension $4 + 3d_L$, to prove Thm.2.3 (1), it's sufficient to show $\dim_{k(x)} T_{X,x} \leq 4 + 3d_L$. As in [13, (4.1)], one has an exact sequence:

$$(7) \quad 0 \rightarrow K(r) \cap T_{X,x} \rightarrow T_{X,x} \xrightarrow{f} \text{Ext}_{\text{Gal}_L}^1(r, r)$$

where $\dim_{k(x)} K(r)$ is a $k(x)$ -vector space of dimension $4 - \dim_{k(x)} \text{End}_{\text{Gal}_L}(r)$. Since

$$(8) \quad \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) = \dim_{k(x)} \text{End}_{\text{Gal}_L}(r) + 4d_L,$$

as in [13, Lem.4.5], it's sufficient to prove

$$(9) \quad \dim_{k(x)} \text{Im}(f) \leq \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - d_L.$$

We view an element \tilde{r} of $\text{Ext}_{\text{Gal}_L}^1(r, r)$ as a rank 2 representation of Gal_L over $k(x)[\epsilon]/\epsilon^2$, whose Sen weights thus have the form $(\text{wt}(\delta_i)_\sigma + \epsilon d_{\sigma,i})_{i=1,2, \sigma \in \Sigma_L}$. We have a $k(x)$ -linear map (the Sen map)

$$\nabla : \text{Ext}_{\text{Gal}_L}^1(r, r) \longrightarrow k(x)^{2d_L}, \quad \tilde{r} \mapsto (d_{\sigma,1}, d_{\sigma,2})_{\sigma \in \Sigma_L},$$

which is known to be surjective (e.g. see [13, Prop.4.9], see also Lem.2.8 below).

For $t \in T_{X,x} : \text{Spec } k(x)[\epsilon]/\epsilon^2 \rightarrow X_{\text{tri}}^\square(\bar{\tau}_L)$, the composition $\text{Spec } k(x)[\epsilon]/\epsilon^2 \rightarrow X_{\text{tri}}^\square(\bar{\tau}_L) \rightarrow X_{\bar{\tau}_L}^\square$ gives a continuous representation $\tilde{r} : \text{Gal}_L \rightarrow \text{GL}_2(k(x)[\epsilon]/\epsilon^2)$, which in fact equals the image of t in $\text{Ext}_{\text{Gal}_L}^1(r, r)$ via f . The composition $\text{Spec } k(x)[\epsilon]/\epsilon^2 \rightarrow X_{\text{tri}}^\square(\bar{\tau}_L) \rightarrow \hat{T}$ gives a character $\tilde{\delta} = \tilde{\delta}_1 \otimes \tilde{\delta}_2 : T(L) \rightarrow E[\epsilon]/\epsilon^2$ satisfying $\tilde{\delta} \equiv \delta \pmod{\epsilon}$. It's known that the Sen weights of \tilde{r} are exactly $(\text{wt}(\tilde{\delta}_1)_\sigma, \text{wt}(\tilde{\delta}_2)_\sigma)_{\sigma \in \Sigma_L} \in (k(x)[\epsilon]/\epsilon^2)^{2d_L}$. The representation \tilde{r} satisfies the following two properties

- (10) $\left\{ \begin{array}{l} \text{(i)} \quad \nabla(\tilde{r}) \in W := \{(d_{\sigma,1}, d_{\sigma,2})_{\sigma \in \Sigma_L} \mid d_{\sigma,1} = d_{\sigma,2}, \forall \sigma \in \Sigma(x)\}; \\ \text{(ii)} \quad \text{there exists an injection of } (\varphi, \Gamma)\text{-modules over } \mathcal{R}_{k(x)[\epsilon]/\epsilon^2} : \mathcal{R}_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}_1) \hookrightarrow D_{\text{rig}}(\tilde{r}); \end{array} \right.$

where (ii) follows from the results in [4] [26, Prop.4.3.5], and (i) is due to Bergdall (cf. [4, Thm.B], see also [11, Lem.9.6]). Indeed, although these references (for (i)) are for crystalline case, an easy variation (of [11, Lem.9.6]) would apply for the case we consider: By (ii), we have an injection

$$\mathcal{R}_{k(x)[\epsilon/\epsilon^2]} \hookrightarrow D_{\text{rig}}(\tilde{r}) \otimes_{\mathcal{R}_{k(x)[\epsilon/\epsilon^2]}} \mathcal{R}_{k(x)[\epsilon/\epsilon^2]}(\tilde{\delta}_1^{-1}) =: D,$$

and $D \pmod{\epsilon} \cong \overline{D} := D_{\text{rig}}(r) \otimes_{\mathcal{R}_{k(x)}} \mathcal{R}_{k(x)}(\delta_1^{-1})$, D has Sen weights $(0, \text{wt}(\tilde{\delta}_2)_\sigma - \text{wt}(\tilde{\delta}_1)_\sigma)_{\sigma \in \Sigma_L}$ and lies in an exact sequence

$$0 \rightarrow \overline{D} \rightarrow D \rightarrow \overline{D} \rightarrow 0.$$

The composition $\mathcal{R}_{k(x)[\epsilon/\epsilon^2]} \hookrightarrow D \rightarrow \overline{D}$ gives a sub- (φ, Γ) -module $\overline{D}' \cong \mathcal{R}_{k(x)}$ of \overline{D} . Since \overline{D} is isomorphic to an extension of $\mathcal{R}_{k(x)}(\delta'_2 \delta_1^{-1})$ by $\mathcal{R}_{k(x)}(\delta'_1 \delta_1^{-1})$, one sees $(\overline{D}')^{\text{sat}} := \overline{D}'[1/t] \cap \overline{D}' \cong \mathcal{R}_{k(x)}(\delta'_1 \delta_1^{-1}) = \mathcal{R}_{k(x)}(\prod_{\sigma \in \Sigma(x)} \sigma^{\text{wt}(\delta)_{2,\sigma} - \text{wt}(\delta)_{1,\sigma}})$. To prove (i), it's sufficient to prove $\text{wt}(\delta)_{2,\sigma} - \text{wt}(\delta)_{1,\sigma}$ is a constant Sen weight for D for all $\sigma \in \Sigma(x)$. Let D' be the preimage of $(\overline{D}')^{\text{sat}}$ in D , which is a saturated sub- (φ, Γ) -module of D , and sits in an exact sequence

$$0 \rightarrow \overline{D} \rightarrow D' \rightarrow (\overline{D}')^{\text{sat}} \rightarrow 0.$$

We claim D' is σ -de Rham for all $\sigma \in \Sigma(x)$ (recall for D' , one can define $D_{\text{dR}}(D')$ which is an $L \otimes_{\mathbb{Q}_p} k(x)$ -module and has a decomposition respectively $D_{\text{dR}}(D') \cong \prod_{\sigma \in \Sigma_L} D_{\text{dR}}(D')_\sigma$, and D' is called σ -de Rham if $\dim_{k(x)} D_{\text{dR}}(D')_\sigma = \text{rk}_{\mathcal{R}_{k(x)}} D'$, we refer to §A for more discussion but in terms of B -pairs), from which we see $\text{wt}(\delta)_{2,\sigma} - \text{wt}(\delta)_{1,\sigma}$ is a constant Sen weight for D' (note \overline{D} has Sen weights $(0, \text{wt}(\delta)_{2,\sigma} - \text{wt}(\delta)_{1,\sigma})_{\sigma \in \Sigma_L}$), and hence a constant Sen weight for D for all $\sigma \in \Sigma(x)$. To prove this claim, it's sufficient to prove the induced map $D_{\text{dR}}(D') \rightarrow D_{\text{dR}}((\overline{D}')^{\text{sat}})$ is surjective (since \overline{D} is σ -de Rham for $\sigma \in \Sigma(x)$). Consider the composition $\mathcal{R}_{k(x)[\epsilon/\epsilon^2]} \hookrightarrow D' \rightarrow \mathcal{R}_{k(x)} \hookrightarrow (\overline{D}')^{\text{sat}}$, where the composition of the first two morphisms is just the natural projection, and induces a projection $D_{\text{dR}}(\mathcal{R}_{k(x)[\epsilon/\epsilon^2]}) \rightarrow D_{\text{dR}}(\mathcal{R}_{k(x)})$, and where the last morphism induces an isomorphism $D_{\text{dR}}(\mathcal{R}_{k(x)}) \xrightarrow{\sim} D_{\text{dR}}((\overline{D}')^{\text{sat}})$. From which, we deduce $D_{\text{dR}}(D') \rightarrow D_{\text{dR}}((\overline{D}')^{\text{sat}})$. The claim follows.

As in [13, §4], we would show the properties (i) (ii) cut off a $k(x)$ -vector subspace of $\text{Ext}_{\text{Gal}_L}^1(r, r)$ of dimension $\dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - d_L$ (from which Thm.2.3 (1) follows). Recall r is trianguline of parameter (δ'_1, δ'_2) :

$$0 \rightarrow \mathcal{R}_{k(x)}(\delta'_1) \rightarrow D_{\text{rig}}(r) \rightarrow \mathcal{R}_{k(x)}(\delta'_2) \rightarrow 0.$$

Consider the composition (where the last one is induced by the natural inclusion $\mathcal{R}_{k(x)}(\delta_1) \hookrightarrow \mathcal{R}_{k(x)}(\delta'_1)$)

$$(11) \quad \begin{aligned} \text{Ext}_{(\varphi, \Gamma)}^1(D_{\text{rig}}(r), D_{\text{rig}}(r)) &\longrightarrow \text{Ext}_{(\varphi, \Gamma)}^1(\mathcal{R}_{k(x)}(\delta'_1), D_{\text{rig}}(r)) \\ &\longrightarrow \text{Ext}_{(\varphi, \Gamma)}^1(\mathcal{R}_{k(x)}(\delta'_1), \mathcal{R}_{k(x)}(\delta'_2)) \xrightarrow{j} \text{Ext}_{(\varphi, \Gamma)}^1(\mathcal{R}_{k(x)}(\delta_1), \mathcal{R}_{k(x)}(\delta'_2)) \end{aligned}$$

One can check \tilde{r} satisfies the property (ii) if and only if $D_{\text{rig}}(\tilde{r})$ lies in the kernel, denoted by V_1 , of the above composition. As in [13, Prop.4.11], one has

Lemma 2.7. $\dim_{k(x)} V_1 = \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - (d_L - |\Sigma(x)|)$.

Proof. We sketch the proof. Since r is very regular, the first two maps in (11) are surjective. Using results in [13, §4.4] (see also the proof of Lem.2.10 below, which is in term of B -pairs), one has $\dim_{k(x)} \text{Im}(j) = d_L - |\Sigma(x)|$, thus

$$\dim_{k(x)} V_1 = \dim_{k(x)} \text{Ext}_{(\varphi, \Gamma)}^1(D_{\text{rig}}(r), D_{\text{rig}}(r)) - (d_L - |\Sigma(x)|) = \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - (d_L - |\Sigma(x)|). \quad \square$$

As in (the proof of) [13, Prop.4.12], one has

Lemma 2.8. *The induced map $\nabla : V_1 \rightarrow k(x)^{2d_L}$ is surjective.*

Proof. This lemma follows from the fact that the trianguline deformations of $D_{\text{rig}}(r)$ over $E[\epsilon]/\epsilon^2$ are contained in V_1 (which is obvious), and the tangent map from the trianguline deformation space to the weight space is surjective. Indeed, for any continuous character $\tilde{\delta}' : L^\times \rightarrow (E[\epsilon]/\epsilon^2)^\times$ with $\tilde{\delta}' \equiv \delta'_1(\delta'_2)^{-1} \pmod{\epsilon}$, consider the exact sequence of (φ, Γ) -modules over $\mathcal{R}_{k(x)}$

$$0 \rightarrow \mathcal{R}_{k(x)}(\delta'_1(\delta'_2)^{-1}) \rightarrow \mathcal{R}_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}') \rightarrow \mathcal{R}_{k(x)}(\delta'_1(\delta'_2)^{-1}) \rightarrow 0,$$

since r is very regular, the induced map $H_{(\varphi, \Gamma)}^1(\mathcal{R}_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}')) \rightarrow H_{(\varphi, \Gamma)}^1(\mathcal{R}_{k(x)}(\delta'_1(\delta'_2)^{-1}))$ is surjective. Let $D' \in H_{(\varphi, \Gamma)}^1(\mathcal{R}_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}'))$ be a preimage of $[D_{\text{rig}}(r) \otimes \mathcal{R}_E \mathcal{R}_E((\delta'_2)^{-1})]$, thus for any continuous character $\tilde{\delta}'_2 : L^\times \rightarrow (k(x)[\epsilon]/\epsilon^2)^\times$ with $\tilde{\delta}'_2 \equiv \delta'_2 \pmod{\epsilon}$, we see $D := D' \otimes_{\mathcal{R}_{k(x)[\epsilon]/\epsilon^2}} \mathcal{R}_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}'_2)$ is a trianguline deformation of $D_{\text{rig}}(r)$ over $\mathcal{R}_{k(x)[\epsilon]/\epsilon^2}$ with Sen weights $(\text{wt}(\tilde{\delta}')_\sigma + \text{wt}(\tilde{\delta}'_2)_\sigma, \text{wt}(\tilde{\delta}'_2)_\sigma)_{\sigma \in \Sigma_L}$. It's straightforward to see $[D] \in V_1$. For any $(a_\sigma, b_\sigma) \in k(x)^{2d_L}$, choose $\tilde{\delta}'$ and $\tilde{\delta}'_2$ such that $\text{wt}(\tilde{\delta}'_2)_\sigma = b_\sigma$, $\text{wt}(\tilde{\delta}')_\sigma = a_\sigma - b_\sigma$, then $\nabla([D]) = (a_\sigma, b_\sigma)_{\sigma \in \Sigma_L}$, so $\nabla|_{V_1}$ is surjective. \square

Put $V := \nabla^{-1}(W)$, thus \tilde{r} satisfies (10) (i) if and only if $\tilde{r} \in V$. So $\text{Im}(f) \subseteq V_1 \cap V$. Since $\nabla|_{V_1}$ is surjective, $\dim_{k(x)} V \cap V_1 = \text{Ext}_{\text{Gal}_L}^1(r, r) - (d_L - |\Sigma(x)|) - |\Sigma(x)| = \text{Ext}_{\text{Gal}_L}^1(r, r) - d_L$. The part (1) of Thm.2.3 follows (cf. (9)), and one gets equalities

$$(12) \quad \begin{cases} \text{Im}(f) = V_1 \cap V, \\ K(r) \cap T_{X,x} = T_{X,x}. \end{cases}$$

By Lem.2.8, the composition

$$(13) \quad T_{X,x} \xrightarrow{f} \text{Im}(f) \xrightarrow{\nabla} W$$

is surjective. Put $W_J := \{(d_{\sigma,1}, d_{\sigma,2})_{\sigma \in \Sigma_L} \mid d_{\sigma,1} = d_{\sigma,2} = 0, \forall \sigma \in J\}$, by definition, $T_{X(\underline{k}_J),x} \subseteq T_{X,x}$ equals the preimage of $W \cap W_J$ via (13), and thus

$$\dim_{k(x)} T_{X(\underline{k}_J),x} = \dim_{k(x)} T_{X,x} - |J \cap \Sigma(x)| - 2|J \cap (\Sigma_L \setminus \Sigma(x))|,$$

Thm.2.3 (2) follows.

We prove Thm.2.5. For $v \in T_{X,x}$, let \tilde{r} be the Gal_L -representation over $k(x)[\epsilon]/\epsilon^2$ associated to $f(v)$ (cf. (7)), by definition and (12), $v \in T_{X_{J-\text{dR}}(\underline{k}_J),x}$ if and only if \tilde{r} satisfies the condition (i) (ii) in (10) and

(iii) \tilde{r} is J -de Rham.

Denote by $\text{Ext}_{\text{Gal}_L, g, J}^1(r, r)$ the $k(x)$ -vector subspace of $\text{Ext}_{\text{Gal}_L}^1(r, r)$ consisting of J -de Rham extensions. By the above discussion, (7) and (12), we have an exact sequence

$$0 \rightarrow K(r) \rightarrow T_{X_{J-\text{dR}}(\underline{k}_J),x} \xrightarrow{f} \text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V_1 \cap V \rightarrow 0.$$

For $J' \subset J$, by definition $v \in T_{X_{J'-\text{dR}}(\underline{k}_J),x}$ if and only if $v \in T_{X_{J'-\text{dR}}(\underline{k}_{J'}),x}$ and $\nabla \circ f(v) \in W_J$, so we have an exact sequence

$$0 \rightarrow K(r) \rightarrow T_{X_{J'-\text{dR}}(\underline{k}_J),x} \xrightarrow{f} \text{Ext}_{\text{Gal}_L, g, J'}^1(r, r) \cap V_1 \cap V \cap \nabla^{-1}(W_J) \rightarrow 0.$$

Thm.2.5 is then an easy consequence of the following proposition.

Proposition 2.9. (1) $\dim_{k(x)} \text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V \cap V_1 = \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - d_L - 2|J|$.

(2) $\dim_{k(x)} \text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V_1 \cap V \cap \nabla^{-1}(W_J) = \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - d_L - 2|J'| - 2|(J \setminus J') \cap (\Sigma_L \setminus \Sigma(x))| - |(J \setminus J') \cap \Sigma(x)|$.

Prop.2.9 would be deduced from the following two lemmas.

Lemma 2.10. $\dim_{k(x)} \text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V_1 = \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - 3|J| - (d_L - |\Sigma(x)| - |J \cap N(\delta')|).$

Lemma 2.11. *The induced map*

$$\nabla : \text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V_1 \longrightarrow W_J$$

is surjective.

Proof of Prop. 2.9. (1) Assuming the above two lemmas, one has thus

$$\begin{aligned} \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) \cap V_1 \cap \nabla^{-1}(W) &= \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) \cap V_1 - |\Sigma(x) \cap (\Sigma_L \setminus J)| \\ &= \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - 3|J| - (d_L - |\Sigma(x)| - |J \cap N(\delta')|) - |\Sigma(x) \cap (\Sigma_L \setminus J)| \\ &= \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - d_L - 3|J| + |J \cap N(\delta')| + |J \cap \Sigma(x)| \\ &= \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - d_L - 2|J|, \end{aligned}$$

where the first equality follows from Lem. 2.11, and the last from $N(\delta') \sqcup \Sigma(x) = N(\delta)$ and $J \subset N(\delta)$.

(2) By (1) applied to J' , the following map is surjective:

$$\text{Ext}_{\text{Gal}_L, g, J'}^1(r, r) \cap V \cap V_1 \longrightarrow W \cap W_{J'}.$$

So (2) follows from the easy fact that $|W \cap W_{J'}| - |W \cap W_J| = 2|(J \setminus J') \cap \Sigma(x)^c| + |(J \setminus J') \cap \Sigma(x)|$. \square

We use the language of B -pairs in the proof of Lem. 2.10 and 2.11.

Proof of Lem. 2.10. Let $W(r)$ denote the $k(x)$ - B -pair associated to r , which lies in an exact sequence of $k(x)$ - B -pairs (cf. App. §A below for notations for B -pairs)

$$0 \rightarrow B_{k(x)}(\delta'_1) \rightarrow W(r) \rightarrow B_{k(x)}(\delta'_2) \rightarrow 0.$$

Identifying $\text{Ext}_{\text{Gal}_L}^1(r, r)$ and $H^1(\text{Gal}_L, W(r) \otimes W(r)^\vee)$, $\text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V_1$ equals thus the kernel of the composition

$$\begin{aligned} (14) \quad H_{g, J}^1(\text{Gal}_L, W(r) \otimes W(r)^\vee) &\longrightarrow H_{g, J}^1(\text{Gal}_L, B_{k(x)}(\delta'_2) \otimes W(r)^\vee) \\ &\longrightarrow H_{g, J}^1(\text{Gal}_L, B_{k(x)}(\delta'_2) \otimes B_{k(x)}((\delta'_1)^{-1})) \cong H_{g, J}^1(\text{Gal}_L, B_{k(x)}(\delta'_2(\delta'_1)^{-1})) \\ &\longrightarrow H_{g, J}^1(\text{Gal}_L, B_{k(x)}(\delta'_2\delta_1^{-1})) \end{aligned}$$

where the first two maps are surjective by Prop. A.5 (since δ is very regular), and the last map is induced by the natural inclusion $B_{k(x)}(\delta'_2(\delta'_1)^{-1}) \hookrightarrow B_{k(x)}(\delta'_2\delta_1^{-1})$. Denote $\delta_0 := \delta'_2\delta_1^{-1}$, $\delta'_0 := \delta'_2(\delta'_1)^{-1}$, thus $\delta_0 = \delta'_0 \prod_{\sigma \in \Sigma(x)} \sigma^{-n_\sigma}$, with $n_\sigma := |\text{wt}(\delta_1)_\sigma - \text{wt}(\delta_2)_\sigma| \in \mathbb{Z}_{\geq 1}$ for $\sigma \in \Sigma_L$, and

$$(15) \quad \text{wt}(\delta'_0)_\sigma = \begin{cases} n_\sigma & \sigma \in \Sigma_L \setminus N(\delta') \\ -n_\sigma & \sigma \in N(\delta') \end{cases}, \quad \text{wt}(\delta_0)_\sigma = \begin{cases} \text{wt}(\delta'_0)_\sigma & \sigma \in \Sigma_L \setminus \Sigma(x) \\ 0 & \sigma \in \Sigma(x) \end{cases}.$$

One has an exact sequence (of Gal_L -complexes)

$$\begin{aligned} 0 \longrightarrow [B_{k(x)}(\delta'_0)_e \oplus B_{k(x)}(\delta'_0)_{\text{dR}}^+ \rightarrow B_{k(x)}(\delta'_0)_{\text{dR}}] \\ \longrightarrow [B_{k(x)}(\delta_0)_e \oplus B_{k(x)}(\delta_0)_{\text{dR}}^+ \rightarrow B_{k(x)}(\delta_0)_{\text{dR}}] \\ \longrightarrow [\oplus_{\sigma \in \Sigma(x)} B_{k(x)}(\delta_0)_{\text{dR}, \sigma}^+ / t^{n_\sigma} \rightarrow 0] \longrightarrow 0 \end{aligned}$$

which induces

$$\begin{aligned} 0 \rightarrow \oplus_{\sigma \in \Sigma(x)} H^0(\text{Gal}_L, B_{k(x)}(\delta_0)_{\text{dR}, \sigma}^+ / t^{n_\sigma}) \rightarrow H^1(\text{Gal}_L, B_{k(x)}(\delta'_0)) \\ \xrightarrow{j} H^1(\text{Gal}_L, B_{k(x)}(\delta_0)) \rightarrow \oplus_{\sigma \in \Sigma(x)} H^1(\text{Gal}_L, B_{k(x)}(\delta_0)_{\text{dR}, \sigma}^+ / t^{n_\sigma}) \rightarrow 0. \end{aligned}$$

By (15), one has $B_{k(x)}(\delta_0)_{\text{dR},\sigma}^+ \cong B_{\text{dR},\sigma}^+$ for $\sigma \in \Sigma(x)$. Thus $\dim_{k(x)} \text{Im}(j) = d_L - |\Sigma(x)|$. Moreover, by [19, (7)], $\text{Im}(j) = H_{g,\Sigma(x)}^1(\text{Gal}_L, B_{k(x)}(\delta_0))$.

We claim that the map j restricts to a surjective map

$$H_{g,J}^1(\text{Gal}_L, B_{k(x)}(\delta'_0)) \longrightarrow H_{g,\Sigma(x) \cup J}^1(\text{Gal}_L, B_{k(x)}(\delta_0)).$$

Indeed we have a commutative diagram

$$\begin{array}{ccc} H^1(\text{Gal}_L, B_{k(x)}(\delta'_0)) & \xrightarrow{j} & H_{g,\Sigma(x)}^1(\text{Gal}_L, B_{k(x)}(\delta_0)) \\ \downarrow & & \downarrow \\ \oplus_{\sigma \in J \cap (\Sigma_L \setminus \Sigma(x))} H^1(\text{Gal}_L, B_{k(x)}(\delta'_0)_{\text{dR},\sigma}^+) & \xrightarrow{\cong} & \oplus_{\sigma \in J \cap (\Sigma_L \setminus \Sigma(x))} H^1(\text{Gal}_L, B_{k(x)}(\delta_0)_{\text{dR},\sigma}^+) \end{array}$$

where the top map is surjective, and the bottom isomorphism follows from (15). Note the kernel of the left vertical map is $H_{g,J \cap (\Sigma_L \setminus \Sigma(x))}^1(\text{Gal}_L, B_{k(x)}(\delta'_0)) \cong H_{g,J}^1(\text{Gal}_L, B_{k(x)}(\delta'_0))$, since for all $\sigma \in J \cap \Sigma(x) \subseteq \Sigma_L \setminus N(\delta')$, $H_{g,\sigma}^1(\text{Gal}_L, B_{k(x)}(\delta'_0)) = H^1(\text{Gal}_L, B_{k(x)}(\delta'_0))$ by [19, Lem.1.11] and (15); and the kernel of the right vertical map is by definition $H_{g,\Sigma(x) \cup J}^1(\text{Gal}_L, B_{k(x)}(\delta_0))$. The claim then follows.

By this claim, the composition (14) induces a surjection

$$H_{g,J}^1(\text{Gal}_L, W(r) \otimes W(r)^\vee) \longrightarrow H_{g,\Sigma(x) \cup J}^1(\text{Gal}_L, B_{k(x)}(\delta_0)).$$

Since δ' is very regular, by Prop.A.3 and Cor.A.4, we have

$$\begin{aligned} \dim_{k(x)} H_{g,J}^1(\text{Gal}_L, W(r) \otimes W(r)^\vee) \\ &= \dim_{k(x)} H^1(\text{Gal}_L, W(r) \otimes W(r)^\vee) - \sum_{\sigma \in J} \dim_{k(x)} H^0(\text{Gal}_L, (W(r) \otimes W(r)^\vee)_{\text{dR},\sigma}^+) \\ &= \dim_{k(x)} \text{Ext}_{\text{Gal}_L}^1(r, r) - 3|J|, \\ \dim_{k(x)} H_{g,\Sigma(x) \cup J}^1(\text{Gal}_L, B_{k(x)}(\delta_0)) &= d_L - |\Sigma(x)| - |J \cap N(\delta')|, \end{aligned}$$

the lemma follows. \square

Proof of Lem.2.11. Let $\tilde{\delta}' : L^\times \rightarrow k(x)[\epsilon]/\epsilon^2$ be a continuous character with $\tilde{\delta}' \equiv \delta'_1(\delta'_2)^{-1} \pmod{\epsilon}$. As discussed in Ex.A.2 (4), the associated $k(x)[\epsilon]/\epsilon^2$ - B -pair $B_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta})$ is σ -de Rham if and only if $\text{wt}(\tilde{\delta}) \in \mathbb{Z}$. For any $\tilde{\delta}' : L^\times \rightarrow k(x)[\epsilon]/\epsilon^2$ as above satisfying moreover $\text{wt}(\tilde{\delta}')_\sigma = \text{wt}(\tilde{\delta})_\sigma$ for all $\sigma \in J$, by Prop.A.5, the natural morphism $B_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}') \rightarrow B_{k(x)}(\delta'_1(\delta'_2)^{-1})$ induces a surjection $H_{g,J}^1(\text{Gal}_L, B_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}')) \rightarrow H_{g,J}^1(\text{Gal}_L, B_{k(x)}(\delta'_1(\delta'_2)^{-1}))$. Let $W' \in H_{g,J}^1(\text{Gal}_L, B_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}'))$ be a preimage of $[W(r) \otimes B_{k(x)}((\delta'_2)^{-1})]$ which is thus J -de Rham, $\tilde{\delta}'_2 : L^\times \rightarrow k(x)[\epsilon]/\epsilon^2$ be a continuous character with $\tilde{\delta}'_2 \equiv \delta'_2 \pmod{\epsilon}$ and $\text{wt}(\tilde{\delta}'_2)_\sigma = \text{wt}(\delta'_2)_\sigma$ for all $\sigma \in J$ (thus $B_{k(x)}(\tilde{\delta}'_2)$ is J -de Rham). It's straightforward to see $W := W' \otimes B_{k(x)[\epsilon]/\epsilon^2}(\tilde{\delta}'_2)$ lies in $\text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V_1$. For any $(a_\sigma, b_\sigma) \in k(x)^{2(d_L - |J|)}$, choose $\tilde{\delta}'$ and $\tilde{\delta}'_2$ as above which satisfy moreover $\text{wt}(\tilde{\delta}'_2)_\sigma = \text{wt}(\delta'_2)_\sigma + b_\sigma \epsilon$, $\text{wt}(\tilde{\delta}')_\sigma = \text{wt}(\delta'_1(\delta'_2)^{-1})_\sigma + (a_\sigma - b_\sigma) \epsilon$ for $\sigma \in J$, then $\nabla([D]) = (a_\sigma, b_\sigma)_{\sigma \in \Sigma_L}$, so $\nabla : \text{Ext}_{\text{Gal}_L, g, J}^1(r, r) \cap V_1 \rightarrow W_J$ is surjective. \square

3. PATCHED EIGENVARIETIES REVISITED

3.1. Setup and notations. We fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Let F^+ be a totally real number field, F be a quadratic imaginary unramified extension of F^+ and c be the unique non-trivial element in $\text{Gal}(F/F^+)$. Suppose for any finite place $v|p$ of F^+ , v is completely decomposed in F . Let E be a finite extension of \mathbb{Q}_p big enough to contain all the embedding of F^+ in $\overline{\mathbb{Q}_p}$, with \mathcal{O}_E its ring of integers and ϖ_E a uniformizer. Denote by Σ_v the set of \mathbb{Q}_p -embeddings of F_v^+ in $\overline{\mathbb{Q}_p}$, $\Sigma_p := \cup_{v|p} \Sigma_v$.

Let G be a quasi-split 2-variables definite unitary group associated to F/F^+ . We fix an isomorphism of algebraic groups $G \times_{F^+} F \cong \mathrm{GL}_2/F$. If v is a finite place of F^+ decomposed in F , \tilde{v} is a place of F dividing v , thus $F_v^+ \cong F_{\tilde{v}}^+$, and we get an isomorphism $i_{\tilde{v}} : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_2(F_{\tilde{v}})$. We fix an open compact subgroup U^p of $G(\mathbb{A}_{F^+}^{\infty,p})$ of the form $U^p = \prod_{v \nmid p} U_v$ where $\mathbb{A}_{F^+}^{\infty,p}$ denotes the finite adeles away from p , and U_v is an open compact subgroup of $G(F_v^+)$. Suppose moreover that U_v is hyperspecial for all the finite places which are inert in F and that (e.g. see the discussion below [13, (3.9)])

$$G(F^+) \cap (hU^p K_p h^{-1}) = \{1\}, \quad \forall h \in G(\mathbb{A}_{F^+}^{\infty})$$

where $K_p := \prod_{v|p} i_{\tilde{v}}^{-1}(\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}}))$ (where \tilde{v} is a place of F above v , note that K_p is independent of the choice of \tilde{v}).

Let S be a finite set of finite places of F^+ containing all the places above p and the places v where U_v is not hyperspecial. Suppose for any $v \in S$, v is completely decomposed in F ; we fix a place \tilde{v} of F above v for $v \in S$ and denote by \tilde{v}^c its image under c . In particular, for $v|p$, we identify $F_{\tilde{v}}$ and F_v^+ , the set $\Sigma_{\tilde{v}}$ of \mathbb{Q}_p -embeddings of $F_{\tilde{v}}$ in $\overline{\mathbb{Q}_p}$ and Σ_v , denote by $F_{\tilde{v},0}$ the maximal unramified extension of \mathbb{Q}_p inside $F_{\tilde{v}}$, $\varpi_{\tilde{v}}$ a uniformizer of $F_{\tilde{v}}$, $d_{\tilde{v}} := [F_{\tilde{v}} : \mathbb{Q}_p]$, $q_{\tilde{v}} := p^{[F_{\tilde{v},0} : \mathbb{Q}_p]}$, $v_{\tilde{v}}(\cdot)$ the additive valuation on $F_{\tilde{v}}$ normalized by sending $\varpi_{\tilde{v}}$ to 1, and $\mathrm{unr}_{\tilde{v}}(z)$ the unramified character of $F_{\tilde{v}}^\times$ sending $\varpi_{\tilde{v}}$ to z .

Suppose moreover for any finite place $v \notin S$ which is decomposed in F , and for any place \tilde{v} dividing v , $U_v = i_{\tilde{v}}^{-1}(\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}}))$, and put \mathbb{T}_v to be the commutative spherical Hecke algebra

$$\mathcal{O}_E[G(F_v^+)/U_v] \xrightarrow{i_{\tilde{v}}} \mathcal{O}_E[\mathrm{GL}_2(F_{\tilde{v}})/\mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})] \cong \mathcal{O}_E[T_{\tilde{v}}, S_{\tilde{v}}^{\pm 1}]$$

where $T_{\tilde{v}} = \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}}) \begin{pmatrix} \varpi_{\tilde{v}} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$ and $S_{\tilde{v}} = \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}}) \begin{pmatrix} \varpi_{\tilde{v}} & 0 \\ 0 & \varpi_{\tilde{v}} \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_{\tilde{v}}})$. One has in fact $i_{\tilde{v}}^{-1}(T_{\tilde{v}}) = i_{\tilde{v}^c}^{-1}(S_{\tilde{v}^c}^{-1} T_{\tilde{v}^c})$, $i_{\tilde{v}}^{-1}(S_{\tilde{v}}) = i_{\tilde{v}^c}^{-1}(S_{\tilde{v}^c})$ with $v = \tilde{v}\tilde{v}^c$. Put $\mathbb{T}^S := \varinjlim_I (\otimes_{v \in I} \mathbb{T}_v)$ for I running over finite sets of finite places of F^+ which are completely decomposed in F , thus \mathbb{T}^S is a commutative \mathcal{O}_E -algebra.

Let $\bar{\rho}$ be a 2-dimensional continuous absolutely irreducible representation of Gal_F over k_E such that $\bar{\rho}^\vee \circ c \cong \bar{\rho} \otimes \bar{\varepsilon}$ and $\bar{\rho}$ is unramified outside S . We associate to $\bar{\rho}$ a maximal ideal $\mathfrak{m}(\bar{\rho})$ of \mathbb{T}^S generated by ϖ_E , $T_{\tilde{v}} - \mathrm{tr}(\bar{\rho}(\mathrm{Frob}_{\tilde{v}}))$ and $\mathrm{Norm}(\tilde{v})S_{\tilde{v}} - \det(\bar{\rho}(\mathrm{Frob}_{\tilde{v}}))$ where $\mathrm{Norm}(\tilde{v})$ denotes the cardinality of the residue field of $F_{\tilde{v}}$, $\mathrm{Frob}_{\tilde{v}}$ denotes a geometric Frobenius. For any \mathbb{T}^S -module M , denote by $M_{\bar{\rho}}$ the localisation of M at $\mathfrak{m}(\bar{\rho})$.

Put $\widehat{S}(U^p, *) := \{f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty})/U^p \rightarrow * \mid f \text{ is continuous}\}$ with $*$ $\in \{E, \mathcal{O}_E\}$, which is equipped with a continuous action of $G(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ by right regular action, and a continuous action of \mathbb{T}^S which commutes with $G(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. Note $\widehat{S}(U^p, E)$ is a E -Banach space with the norm defined by the lattice $\widehat{S}(U^p, \mathcal{O}_E)$. In fact $\widehat{S}(U^p, E)$ is an admissible unitary Banach representation of $G(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$.

Recall the automorphic representations of $G(\mathbb{A}_{F^+})$ are the irreducible constituents of the \mathbb{C} -vector space of functions $f : G(F^+) \backslash G(\mathbb{A}_{F^+}) \rightarrow \mathbb{C}$, which are

- \mathcal{C}^∞ when restricted to $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$,
- locally constant when restricted to $G(\mathbb{A}_{F^+}^{\infty})$,
- $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ -finite,

where $G(\mathbb{A}_{F^+})$ acts on this space via right translation. An automorphic representation π is isomorphic to $\pi_\infty \otimes_{\mathbb{C}} \pi^\infty$ where $\pi_\infty = W_\infty$ is an irreducible algebraic representation of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ over \mathbb{C} and $\pi^\infty \cong \mathrm{Hom}_{G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})}(W_\infty, \pi) \cong \otimes'_v \pi_v$ is an irreducible smooth representation of $G(\mathbb{A}_{F^+}^{\infty})$. The algebraic representation W_∞ is defined over $\overline{\mathbb{Q}}$ via ι_∞ , and we denote by W_p its base change to $\overline{\mathbb{Q}_p}$, which is thus an irreducible algebraic representation of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ over $\overline{\mathbb{Q}_p}$. Via the decomposition $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$

$\mathbb{Q}_p) \xrightarrow{\sim} \prod_{v \in \Sigma_p} G(F_v^+)$, one has $W_p \cong \otimes_{v \in \Sigma_p} W_v$ where W_v is an irreducible algebraic representation of $G(F_v^+)$. One can also prove π^∞ is defined over a number field via ι_∞ (e.g. see [2, §6.2.3]). Denote by $\pi^{\infty, p} := \otimes'_{v \nmid p} \pi_v$, thus $\pi \cong \pi^{\infty, p} \otimes_{\overline{\mathbb{Q}}} \pi_p$. Let $m(\pi) \in \mathbb{Z}_{\geq 1}$ be the multiplicity of π in the space of functions as above.

Proposition 3.1 ([11, Prop.5.1]). *One has a $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times \mathbb{T}^S$ -invariant isomorphism*

$$\widehat{S}(U^p, E)^{\text{alg}} \otimes_E \overline{\mathbb{Q}_p} \cong \bigoplus_{\pi} \left((\pi^{\infty, p})^{U^p} \otimes_{\overline{\mathbb{Q}}} (\pi_p \otimes_{\overline{\mathbb{Q}}} W_p) \right)^{\oplus m(\pi)},$$

where $\widehat{S}(U^p, E)^{\text{alg}}$ denotes the locally algebraic subrepresentation of $\widehat{S}(U^p, E)$, $\pi \cong \pi_\infty \otimes_{\mathbb{C}} \pi^\infty$ runs through the automorphic representation of $G(\mathbb{A}_{F^+})$ and W_p is associated to π_∞ as above.

Let $G_p := \prod_{v|p} \text{GL}_2(F_{\bar{v}})$ (which is isomorphic to $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$), $B_p := \prod_{v|p} B(F_{\bar{v}})$ with $B(F_{\bar{v}})$ the Borel subgroup of upper triangular matrices, \overline{B}_p the opposite of B_p , $N_p \cong \prod_{v|p} N(F_{\bar{v}})$ the unipotent radical of B_p , and $T_p \cong \prod_{v|p} T(F_{\bar{v}}) =: \prod_{v|p} T_{\bar{v}}$ the Levi subgroup of B_p . For a closed subgroup H of G_p , denote by $H^0 := H \cap K_p$. Denote by $\mathfrak{g}_p, \mathfrak{b}_p, \overline{\mathfrak{b}}_p, \mathfrak{n}_p, \mathfrak{t}_p, \mathfrak{g}_{\bar{v}}, \mathfrak{b}_{\bar{v}}, \overline{\mathfrak{b}}_{\bar{v}}, \mathfrak{n}_{\bar{v}}, \mathfrak{t}_{\bar{v}}$ the Lie algebra of $G_p, B_p, \overline{B}_p, N_p, T_p, \text{GL}_2(F_{\bar{v}}), B(F_{\bar{v}}), \overline{B}(F_{\bar{v}}), N(F_{\bar{v}}), T(F_{\bar{v}})$ respectively. Recall one has an isomorphism $\mathfrak{g}_p \otimes_{\mathbb{Q}_p} E \cong \prod_{v|p} \prod_{\sigma \in \Sigma_{\bar{v}}} \mathfrak{g}_{\bar{v}} \otimes_{F_{\bar{v}}, \sigma} E$, for $J \subseteq \Sigma_p$, put $\mathfrak{g}_J := \prod_{v|p} \prod_{\sigma \in J_{\bar{v}}} \mathfrak{g}_{\bar{v}} \otimes_{F_{\bar{v}}, \sigma} E$ with $J_{\bar{v}} := J \cap \Sigma_{\bar{v}}$. Similarly, we get Lie algebras over E : $\mathfrak{b}_J, \overline{\mathfrak{b}}_J$ etc.

A weight of $\mathfrak{t}_p \otimes_{\mathbb{Q}_p} E$ (with values in \overline{E}) would be denoted by $\underline{\lambda}_{\Sigma_p} = (\lambda_{1, \sigma}, \lambda_{2, \sigma})_{\sigma \in \Sigma_p} \in \overline{E}^{2|\Sigma_p|}$ with

$$\underline{\lambda}_{\Sigma_p} \left(\prod_{\sigma \in \Sigma_p} \text{diag}(a_\sigma, d_\sigma) \right) = \sum_{\sigma \in \Sigma_p} a_\sigma \lambda_{1, \sigma} + d_\sigma \lambda_{2, \sigma}.$$

For $J \subseteq \Sigma_p$, let $\underline{\lambda}_J := (\lambda_{1, \sigma}, \lambda_{2, \sigma})_{\sigma \in J}$, which would also be viewed as a weight of $\mathfrak{t}_p \otimes_{\mathbb{Q}_p} E$ via

$$\underline{\lambda}_J \left(\prod_{\sigma \in \Sigma_p} \text{diag}(a_\sigma, d_\sigma) \right) = \sum_{\sigma \in J} a_\sigma \lambda_{1, \sigma} + d_\sigma \lambda_{2, \sigma}.$$

Recall $\underline{\lambda}_J$ is dominant for \mathfrak{b}_p if $\lambda_{1, \sigma} - \lambda_{2, \sigma} \in \mathbb{Z}_{\geq 0}$ for all $\sigma \in J$. If $\underline{\lambda}_J$ is integral, i.e. $\lambda_{i, \sigma} \in \mathbb{Z}$ for all $\sigma \in J$, and dominant, then there exists a unique irreducible algebraic (and locally J -analytic) representation $L(\underline{\lambda}_J)$ of G_p over E with highest weight $\underline{\lambda}_J$ and one has $L(\underline{\lambda}_J) \cong \otimes_{\sigma \in J} L(\underline{\lambda}_{\{\sigma\}})$. For a locally \mathbb{Q}_p -analytic representation V of G_p , $\underline{\lambda}_J$ integral and dominant, put

$$(16) \quad V(\underline{\lambda}_J) := (V \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J\text{-an}} \otimes_E L(\underline{\lambda}_J)$$

(where $L(\underline{\lambda}_J)'$ denotes the dual of $L(\underline{\lambda}_J)$), “ $\Sigma_p \setminus J\text{-an}$ ” denotes the locally $\Sigma_p \setminus J$ -analytic vectors, see §B below) which is in fact a subrepresentation of V .

For a locally \mathbb{Q}_p -analytic character $\delta = \delta_1 \otimes \delta_2$ of T_p over E , let $\delta_{\bar{v}} = \delta_{1, \bar{v}} \otimes \delta_{2, \bar{v}} := \delta|_{T_{\bar{v}}}$ for $v|p$. Let

$$\text{wt}(\delta) = (\text{wt}(\delta)_{1, \sigma}, \text{wt}(\delta)_{2, \sigma})_{\sigma \in \Sigma_p} := (\text{wt}(\delta_1)_\sigma, \text{wt}(\delta_2)_\sigma)_{\sigma \in \Sigma_p}$$

be the induced weight of $\mathfrak{t}_p \otimes_{\mathbb{Q}_p} E$. For an integral weight $\underline{\lambda}_{\Sigma_p}$ of $\mathfrak{t}_p \otimes_{\mathbb{Q}_p} E$, denote by $\delta_{\underline{\lambda}_{\Sigma_p}}$ the algebraic character of T_p with weight $\underline{\lambda}_{\Sigma_p}$, i.e. $\delta_{\underline{\lambda}_{\Sigma_p}} = \prod_{\sigma \in \Sigma_p} \sigma^{\lambda_{1, \sigma}} \otimes \sigma^{\lambda_{2, \sigma}}$.

Denote by \widehat{T}_p (resp. $\widehat{T}_{\bar{v}}$) the rigid space over E parameterizing locally \mathbb{Q}_p -analytic characters of T_p (resp. of $T_{\bar{v}}$), thus $\widehat{T}_p \cong \prod_{v|p} \widehat{T}_{\bar{v}}$. For $J \subset \Sigma_p$ (resp. $J_{\bar{v}} := J \cap \Sigma_{\bar{v}}$), denote by $\widehat{T}_{p, J}$ (resp. $\widehat{T}_{\bar{v}, J_{\bar{v}}}$) the rigid closed subspace of \widehat{T}_p (resp. of $\widehat{T}_{\bar{v}}$) parameterizing locally J -analytic (resp. locally $J_{\bar{v}}$ -analytic) characters of T_p (resp. of $T_{\bar{v}}$), thus $\widehat{T}_{p, J} \cong \prod_{v|p} \widehat{T}_{\bar{v}, J_{\bar{v}}}$. The rigid space $\widehat{T}_{\bar{v}, J_{\bar{v}}}$ is smooth and equidimensional of dimension $|J_{\bar{v}}|$, thus $\widehat{T}_{p, J}$ is smooth and equidimensional of dimension $|J|$. Indeed, one has an étale morphism (of infinite degree)

$$(17) \quad \widehat{T}_{\bar{v}} \longrightarrow (\mathbb{A}^1 \times \mathbb{A}^1)^{|\Sigma_{\bar{v}}|}, \quad \chi \mapsto \text{wt}(\chi),$$

and $\widehat{T}_{\bar{v}, J_{\bar{v}}}$ is just the preimage of $\{(\lambda_{1,\sigma}, \lambda_{2,\sigma})_{\sigma \in \Sigma_{\bar{v}}} \mid \lambda_{i,\sigma} = 0, \forall \sigma \in J_{\bar{v}}, i = 1, 2\}$. More generally, let $\underline{\lambda}_J = (\lambda_{1,\sigma}, \lambda_{2,\sigma})_{\sigma \in J} \in E^{2|J|}$, $\underline{\lambda}_{J_{\bar{v}}} := (\lambda_{1,\sigma}, \lambda_{2,\sigma})_{\sigma \in J_{\bar{v}}} \in E^{2|J_{\bar{v}}|}$ for $v|p$, denote by $\widehat{T}_{\bar{v}}(\underline{\lambda}_{J_{\bar{v}}})$ the preimage of $\{(\lambda'_{1,\sigma}, \lambda'_{2,\sigma})_{\sigma \in \Sigma_{\bar{v}}} \mid \lambda'_{i,\sigma} = \lambda_{i,\sigma}, \forall \sigma \in J_{\bar{v}}, i = 1, 2\}$ via (17), and put $\widehat{T}_p(\underline{\lambda}_J) := \prod_{v|p} \widehat{T}_{\bar{v}}(\underline{\lambda}_{J_{\bar{v}}})$. Note for any locally \mathbb{Q}_p -analytic character δ of T_p over E with $\text{wt}(\delta)_{i,\sigma} = \lambda_{i,\sigma}$ for $\sigma \in J, i = 1, 2$, the isomorphism

$$\widehat{T}_p \xrightarrow{\sim} \widehat{T}_p, \delta' \mapsto \delta' \delta,$$

induces an isomorphism $\widehat{T}_{p,J} \xrightarrow{\sim} \widehat{T}_p(\underline{\lambda}_J)$. Let $T_p^0 := K_p \cap T_p$, and denote by \widehat{T}_p^0 the rigid space over E parameterizing locally \mathbb{Q}_p -analytic characters of T_p^0 , thus the restriction map induces a projection

$$\widehat{T}_p \twoheadrightarrow \widehat{T}_p^0.$$

Denote by $\widehat{T}_{p,J}^0, \widehat{T}_{\bar{v}}^0, \widehat{T}_{\bar{v}, J_{\bar{v}}}^0, \widehat{T}_p^0(\underline{\lambda}_J), \widehat{T}_{\bar{v}}^0(\underline{\lambda}_{J_{\bar{v}}})$ the image in \widehat{T}_p^0 of $\widehat{T}_{p,J}, \widehat{T}_{\bar{v}}, \widehat{T}_{\bar{v}, J_{\bar{v}}}, \widehat{T}_p(\underline{\lambda}_J), \widehat{T}_{\bar{v}}(\underline{\lambda}_{J_{\bar{v}}})$ respectively.

Let V be an E -vector space equipped with an E -linear action of A (with A a set of operators), χ a system of eigenvalues of A , denote by $V[A = \chi]$ the χ -eigenspace, $V\{A = \chi\}$ the generalized χ -eigenspace. If A is moreover an \mathcal{O}_E -algebra with an ideal \mathfrak{J} , denote by $V[\mathfrak{J}]$ the subspace of vectors killed by \mathfrak{J} , and $V\{\mathfrak{J}\} := \varinjlim_n V[\mathfrak{J}^n]$.

3.2. Eigenvarieties. We briefly recall some properties of the eigenvariety of G (with tame level U^p).

Let $\bar{\rho}$ be a 2-dimensional continuous absolutely irreducible representation $\bar{\rho}$ of Gal_F over k_E such that $\bar{\rho}^\vee \circ c \cong \bar{\rho} \otimes \bar{\epsilon}$, $\bar{\rho}$ is unramified outside S and $\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{alg}} \neq 0$. Let $J \subseteq \Sigma_p$, $\underline{\lambda}_J = (\lambda_{1,\sigma}, \lambda_{2,\sigma})_{\sigma \in J} \in \mathbb{Z}^{2|J|}$ with $\lambda_{1,\sigma} \geq \lambda_{2,\sigma}$ for all $\sigma \in J$. Consider $\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{an}}(\underline{\lambda}_J)$, which is an admissible locally \mathbb{Q}_p -analytic representation of G_p equipped with a continuous action of \mathbb{T}^S which commutes with G_p . Note $\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{an}}(\underline{\lambda}_J)$ is in fact a closed subrepresentation of $\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{an}}$. Applying the Jacquet-Emerton functor ([22]), we get an essentially admissible locally \mathbb{Q}_p -analytic representation of T_p : $J_{B_p}(\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{an}}(\underline{\lambda}_J))$, which is also equipped with a continuous action of \mathbb{T}^S commuting with T_p . By [20, §6.4], to $J_{B_p}(\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{an}}(\underline{\lambda}_J))$, is naturally associated a coherent sheaf $\mathcal{M}(U^p, \underline{\lambda}_J)_{\bar{\rho}}$ over \widehat{T}_p such that

$$\Gamma(\mathcal{O}(\widehat{T}_p), \mathcal{M}(U^p, \underline{\lambda}_J)_{\bar{\rho}}) \xrightarrow{\sim} J_{B_p}(\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{an}}(\underline{\lambda}_J))'$$

where $\Gamma(X, \mathcal{F})$ denotes the sections of a sheaf \mathcal{F} on a rigid space X , $(\cdot)'$ denotes the continuous dual equipped with strong topology. This association is functorial, so $\mathcal{M}(U^p, \underline{\lambda}_J)_{\bar{\rho}}$ is equipped with an $\mathcal{O}(\widehat{T}_p)$ -linear action of \mathbb{T}^S . Using Emerton's method (cf. [21, §2.3]), we can construct an eigenvariety from the triplet $\{\mathcal{M}(U^p, \underline{\lambda}_J)_{\bar{\rho}}, \mathcal{O}(\widehat{T}_p), \mathbb{T}^S\}$:

Theorem 3.2. *There exists a rigid space $\mathcal{E}(U^p, \underline{\lambda}_J)_{\bar{\rho}}$ over E finite over \widehat{T}_p and equipped with a morphism*

$$\mathcal{O}(\widehat{T}_p) \otimes_{\mathcal{O}_E} \mathbb{T}^S \longrightarrow \mathcal{O}(\mathcal{E}(U^p, \underline{\lambda}_J)_{\bar{\rho}})$$

such that

- (1) *a closed point of $\mathcal{E}(U^p, \underline{\lambda}_J)_{\bar{\rho}}$ is determined by its induced closed point $\delta : T_p \rightarrow \overline{E}^\times$ of \widehat{T}_p and its induced system of eigenvalues $\mathfrak{h} : \mathbb{T}^S \rightarrow \overline{E}$, and would be denoted by (\mathfrak{h}, δ) ;*
- (2) *$(\mathfrak{h}, \delta) \in \mathcal{E}(U^p, \underline{\lambda}_J)_{\bar{\rho}}(\overline{E})$ if and only if the eigenspace*

$$(J_{B_p}(\widehat{S}(U^p, E)_{\bar{\rho}}^{\text{an}}(\underline{\lambda}_J)_{\bar{\rho}}) \otimes_E \overline{E})[\mathbb{T}^S = \mathfrak{h}, T_p = \delta] \neq 0.$$

Remark 3.3. *By definition (and [18, Lem.6.2.12]),*

$$J_{B_p}(\widehat{S}(U^p, E)_{\bar{\rho}}(\underline{\lambda}_J)) \cong J_{B_p}((\widehat{S}(U^p, E)^{\text{an}} \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J - \text{an}}) \otimes_E \delta_{\underline{\lambda}_J},$$

from which we deduce $\mathcal{M}(U^p, \underline{\lambda}_J)_{\bar{\rho}}$ is supported on $\widehat{T}_p(\underline{\lambda}_J)$ and hence the natural morphism $\mathcal{E}(U^p, \underline{\lambda}_J)_{\bar{\rho}} \rightarrow \widehat{T}_p$ factors through $\widehat{T}_p(\underline{\lambda}_J)$.

Recall a closed point $z = (\mathfrak{h}, \delta)$ of $\mathcal{E}_0(U^p, \underline{\lambda}_J)_{\overline{p}}$ is called classical if

$$(J_{B_p}(\widehat{S}(U^p, E)_{\overline{p}}(\underline{\lambda}_J)^{\text{alg}}) \otimes_E \overline{E})[\mathbb{T}^S = \mathfrak{h}, T_p = \delta] \neq 0.$$

Theorem 3.4. *The rigid space $\mathcal{E}(U^p, \underline{\lambda}_J)_{\overline{p}}$ is equidimensional of dimension $2|\Sigma_p \setminus J|$, and the set of classical points is an accumulation Zariski-dense subset of $\mathcal{E}(U^p, \underline{\lambda}_J)_{\overline{p}}(\overline{E})$.*

Remark 3.5. *The proof of the theorem is omitted, since it's an easy variation of that in the patched eigenvariety case given below (and the same as in the case of eigenvarieties for unitary Shimura curves [18, Prop.6.2.30]), where there are two key points:*

- (1) a classicality result as Prop.3.15 (see also [18, Cor.6.2.28]);
- (2) there exists an open compact normal subgroup H of G_p such that $\widehat{S}(U^p, E)^{\text{an}}|_H \cong \mathcal{C}^{\mathbb{Q}_p - \text{an}}(H, E)$ as locally \mathbb{Q}_p -analytic representations of H , where the latter denotes the space of locally \mathbb{Q}_p -analytic functions of H equipped with the right regular action of H ; this fact in particular allows [22, Prop.4.2.36] to apply.

Put $\mathcal{E}(U^p)_{\overline{p}} := \mathcal{E}(U^p, \lambda_{\emptyset})_{\overline{p}}$. The following proposition follows easily from the natural $G_p \times \mathbb{T}^S$ -invariant injection $\widehat{S}(U^p, E)^{\text{an}}(\underline{\lambda}_J) \hookrightarrow \widehat{S}(U^p, E)^{\text{an}}$.

Proposition 3.6. *There exists a natural closed embedding*

$$\mathcal{E}(U^p, \underline{\lambda}_J)_{\overline{p}} \hookrightarrow \mathcal{E}(U^p)_{\overline{p}}, \quad (\mathfrak{h}, \delta) \mapsto (\mathfrak{h}, \delta)_{\overline{p}}.$$

By discussion in Rem.3.3, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}(U^p, \underline{\lambda}_J)_{\overline{p}} & \longrightarrow & \mathcal{E}(U^p)_{\overline{p}} \\ \downarrow & & \downarrow \\ \widehat{T}_p(\underline{\lambda}_J) & \longrightarrow & \widehat{T}_p \end{array}.$$

Note this diagram should *not* be cartesian in general. Indeed, as we would see later in patched eigenvariety case, the difference between $\mathcal{E}(U^p, \underline{\lambda}_J)$ and $\mathcal{E}(U^p, \underline{\lambda}_J)' := \mathcal{E}(U^p)_{\overline{p}} \times_{\widehat{T}_p} \widehat{T}_p(\underline{\lambda}_J)$ somehow would be the key point for the existence of companion points.

Recall there exists a family of Gal_F -representations on $\mathcal{E}(U^p)_{\overline{p}}$, in particular, for any closed point $z = (\mathfrak{h}, \delta)$ of $\mathcal{E}(U^p)_{\overline{p}}$, there exists a 2-dimensional continuous representation ρ_z of Gal_F over $k(z)$, which is unramified outside S and satisfies

$$\rho_z(\text{Frob}_{\tilde{v}})^2 - \mathfrak{h}(T_{\tilde{v}}) \text{Frob}_{\tilde{v}} + \text{Norm}(\tilde{v}) \mathfrak{h}(S_{\tilde{v}}) = 0$$

for $v \notin S$ completely decomposed in F , and $\tilde{v}|v$, $\text{Frob}_{\tilde{v}} \in \text{Gal}_{F_{\tilde{v}}}$ is a geometric Frobenius. The (semi-simplification of the) reduction of ρ_z modulo $\varpi_{k(z)}$ (a uniformizer of $k(z)$) is isomorphic to $\overline{\rho}$.

Theorem 3.7. *Keep the above notation, for $v|p$, the restriction $\rho_{z, \tilde{v}} := \rho_z|_{\text{Gal}_{F_{\tilde{v}}}}$ is trianguline. If $z \in \mathcal{E}(U^p, \underline{\lambda}_J)_{\overline{p}}$ for $J \subseteq \Sigma_p$ (which implies $\text{wt}(\delta)_{i, \sigma} = \lambda_{i, \sigma}$ for $\sigma \in J$, $i = 1, 2$), then for $v|p$, $\rho_{z, \tilde{v}}$ is moreover $J_{\tilde{v}}$ -de Rham of Hodge-Tate weights $(-\lambda_{1, \sigma}, 1 - \lambda_{2, \sigma})$ for $\sigma \in J_{\tilde{v}}$ where $J_{\tilde{v}} := J \cap \Sigma_{\tilde{v}}$.*

Remark 3.8. *By Thm.3.4, the first part of Thm.3.7 follows from the global triangulation theory ([25], [26]) applied to $\mathcal{E}(U^p)_{\overline{p}}$; and the second part follows from results of Shah [31] applied to $\mathcal{E}(U^p, \underline{\lambda}_J)_{\overline{p}}$.*

In particular, $\mathcal{E}(U^p, \underline{\lambda}_J)_{\overline{p}}$ gives a $J_{\tilde{v}}$ -de Rham family for $v|p$ while $\mathcal{E}(U^p, \underline{\lambda}_J)'_{\overline{p}}$ should *only* give a $J_{\tilde{v}}$ -Hodge-Tate family for $v|p$. The picture would be more clear in the patched eigenvariety case.

3.3. Patched eigenvarieties.

3.3.1. *Patched Banach representation.* Denote by $R_{\bar{\rho}, S}$ the deformation ring (which is a complete noetherian local \mathcal{O}_E -algebra with residue field k_E) which pro-represents the functor associating to local artinian \mathcal{O}_E -algebras A the sets of isomorphism classes of deformations ρ_A of $\bar{\rho}$ over A such that ρ_A is unramified outside S and $\rho_A^\vee \circ c \cong \rho_A \otimes \varepsilon$. For a finite place \tilde{v} of F , put $\bar{\rho}_{\tilde{v}} := \bar{\rho}|_{\text{Gal}_{F_{\tilde{v}}}}$, and $R_{\bar{\rho}_{\tilde{v}}}^\square$ to be the maximal reduced and p -torsion free quotient of $R_{\bar{\rho}_{\tilde{v}}}^\square$, $R_{\bar{\rho}_S}^\square := \widehat{\otimes}_{v \in S} R_{\bar{\rho}_v}^\square$. Fix $g \in \mathbb{Z}_{\geq 1}$, let $R_\infty := R_{\bar{\rho}_S}^\square[[x_1, \dots, x_g]]$, $S_\infty := \mathcal{O}[[y_1, \dots, y_q]]$ with $q = g + 2[F^+ : \mathbb{Q}] + 4|S|$, and $\mathfrak{a} = (y_1, \dots, y_q) \subset S_\infty$.

Suppose moreover $p > 2$ and $\bar{\rho}|_{\text{Gal}_{F(\zeta_p)}}$ adequate (cf. [32]) in the following, by [32, Prop.6.7], $\widehat{S}(U^p, E)_{\bar{\rho}}$ is naturally equipped with a $R_{\bar{\rho}, S}$ -action. Moreover, the action of $R_{\bar{\rho}, S}$ on $\widehat{S}(U^p, E)_{\bar{\rho}}$ factors through $R_{\bar{\rho}, S} \twoheadrightarrow R_{\bar{\rho}, S}$, where $R_{\bar{\rho}, S}$ is the deformation ring associated to the deformation problem (as in [17, §2.3], and we use the notation of *loc. cit.*)

$$S = (F/F^+, S, \tilde{S}, \mathcal{O}_E, \bar{\rho}, \varepsilon^{-1} \delta_{F/F^+}^2, \{R_{\bar{\rho}_v}^\square\}_{v \in S})$$

where $\tilde{S} = \{\tilde{v} \mid v \in S\}$, and δ_{F/F^+} is the quadratic character of Gal_{F^+} associated to the extension F/F^+ .

By [14] (see also [12, Thm.3.5]), we have

- (1) a continuous R_∞ -admissible unitary representation Π_∞ of G_p over E together with a G_p -stable and R_∞ -stable unit ball $\Pi_\infty^\circ \subset \Pi_\infty$;
- (2) a morphism of local \mathcal{O}_E -algebras $S_\infty \rightarrow R_\infty$ such that $M_\infty := \text{Hom}_{\mathcal{O}_L}(\Pi_\infty^\circ, \mathcal{O}_E)$ is finite projective as $S_\infty[[K_p]]$ -module;
- (3) an isomorphism $R_\infty/\mathfrak{a}R_\infty \cong R_{\bar{\rho}, S}$ and a $G_p \times R_\infty/\mathfrak{a}R_\infty$ -invariant isomorphism $\Pi_\infty[\mathfrak{a}] \cong \widehat{S}(U^p, E)_{\bar{\rho}}$, where R_∞ acts on $\widehat{S}(U^p, E)_{\bar{\rho}}$ via $R_\infty/\mathfrak{a}R_\infty \cong R_{\bar{\rho}, S}$.

3.3.2. *Patched eigenvariety and some stratifications.* Recall (cf. [12, §3.1]) for an R_∞ -admissible representation Π of G_p over E , a vector $v \in \Pi$ is called *locally R_∞ -analytic* if it's locally \mathbb{Q}_p -analytic for the action of $\mathbb{Z}_p^s \times G_p$ with respect to a presentation $\mathcal{O}_E[[\mathbb{Z}_p^s]] \twoheadrightarrow R_\infty$. And it's shown in *loc. cit.*, this definition is independent of the choice of the presentation. As in *loc. cit.*, denote by $\Pi^{R_\infty\text{-an}}$ the subspace of locally R_∞ -analytic vectors.

Let $J \subseteq \Sigma_p$, $\underline{\lambda}_J := (\lambda_{1,\sigma}, \lambda_{2,\sigma})_{\sigma \in J} \in \mathbb{Z}^{2|J|}$, and suppose $\lambda_{1,\sigma} \geq \lambda_{2,\sigma}$ for all $\sigma \in J$. Consider $\Pi_\infty^{R_\infty\text{-an}}(\underline{\lambda}_J)$ (cf. (16)), which is a locally \mathbb{Q}_p -analytic $U(\mathfrak{g}_J)$ -finite representation of G_p (cf. §App.B), stable under R_∞ and is moreover admissible as a locally \mathbb{Q}_p -analytic representation of $G_p \times \mathbb{Z}_p^s$ (where the action \mathbb{Z}_p^s is induced by that of R_∞ via any presentation $\mathcal{O}_E[[\mathbb{Z}_p^s]] \twoheadrightarrow R_\infty$). In fact, $\Pi_\infty^{R_\infty\text{-an}}(\underline{\lambda}_J)$ is a closed subrepresentation of $\Pi_\infty^{R_\infty\text{-an}}$ by Prop.B.1 (where the closedness follows from the same argument as in Cor.B.2). Note also that, for $J' \subseteq J$, by Lem.B.3 (3), $\Pi_\infty^{R_\infty\text{-an}}(\underline{\lambda}_J)$ is a closed subrepresentation of $\Pi_\infty^{R_\infty\text{-an}}(\underline{\lambda}_{J'})$.

Applying Jacquet-Emerton functor, we get a locally \mathbb{Q}_p -analytic representation $J_{B_p}(\Pi_\infty^{R_\infty\text{-an}}(\underline{\lambda}_J))$ of T_p equipped with a continuous action of R_∞ , which is moreover essentially admissible as a locally \mathbb{Q}_p -analytic representation of $T_p \times \mathbb{Z}_p^s$. Let $\mathfrak{X}_\infty := (\text{Spf } R_\infty)^{\text{rig}}$, $R_\infty^{\text{rig}} := \mathcal{O}(\mathfrak{X}_\infty)$, the strong dual $J_{B_p}(\Pi_\infty^{R_\infty\text{-an}}(\underline{\lambda}_J))'$ is thus a coadmissible $R_\infty^{\text{rig}} \widehat{\otimes}_E \mathcal{O}(\widehat{T}_p)$ -module, which corresponds to a coherent sheaf $\mathcal{M}_\infty(\underline{\lambda}_J)$ over $\mathfrak{X}_\infty \times \widehat{T}_p$ such that

$$\Gamma(\mathfrak{X}_\infty \times \widehat{T}_p, \mathcal{M}_\infty(\underline{\lambda}_J)) \cong J_{B_p}(\Pi_\infty(\underline{\lambda}_J))'.$$

Let $X_p(\bar{\rho}, \underline{\lambda}_J)$ be the support of $\mathcal{M}_\infty(\underline{\lambda}_J)$ on $\mathfrak{X}_\infty \times \widehat{T}_p$, in particular, for $x = (y, \delta) \in \mathfrak{X}_\infty \times \widehat{T}_p$, $x \in X_p(\bar{\rho}, \underline{\lambda}_J)$ if and only if the corresponding eigenspace

$$J_{B_p}(\Pi_\infty^{R_\infty\text{-an}}(\underline{\lambda}_J))[\mathfrak{p}_y, T_p = \delta] \neq 0,$$

where \mathfrak{p}_y denote the maximal ideal of $R_\infty[\frac{1}{p}]$ corresponding to y . Since

$$(18) \quad J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\lambda}_J)) \cong J_{B_p}((\Pi_\infty^{R_\infty-\text{an}} \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J - \text{an}}) \otimes_E \delta_{\underline{\lambda}_J},$$

it's straightforward to see the action of $\mathcal{O}(\widehat{T}_p)$ on $J_{B_p}(\Pi_\infty^{R_\infty-\text{an}})$ factors through $\mathcal{O}(\widehat{T}_p(\underline{\lambda}_J))$, and hence $\mathcal{M}_\infty(\underline{\lambda}_J)$ is supported on $\mathfrak{X}_\infty \times \widehat{T}_p(\underline{\lambda}_J)$. So the natural injection $X_p(\bar{\rho}, \underline{\lambda}_J) \hookrightarrow \mathfrak{X}_\infty \times \widehat{T}_p$ factors through $\mathfrak{X}_\infty \times \widehat{T}_p(\underline{\lambda}_J)$.

Let $J' \subseteq J$, one has a natural projection of coadmissible $\mathcal{O}(\mathfrak{X}_\infty \times \widehat{T}_p)$ -modules $\mathcal{M}(\bar{\rho}, \underline{\lambda}_{J'}) \twoheadrightarrow \mathcal{M}(\bar{\rho}, \underline{\lambda}_J)$ induced by the natural inclusion $J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\lambda}_J)) \hookrightarrow J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\lambda}_{J'}))$. Consequently, we see $X_p(\bar{\rho}, \underline{\lambda}_J)$ is naturally a rigid closed subspace of $X_p(\bar{\rho}, \underline{\lambda}_{J'})$, and we have a commutative diagram

$$\begin{array}{ccc} X_p(\bar{\rho}, \underline{\lambda}_J) & \longrightarrow & X_p(\bar{\rho}, \underline{\lambda}_{J'}) \\ \downarrow & & \downarrow \\ \widehat{T}_p(\underline{\lambda}_J) & \longrightarrow & \widehat{T}_p(\underline{\lambda}_{J'}) \end{array}.$$

Let $X_p(\bar{\rho}) := X_p(\bar{\rho}, \underline{\lambda}_\emptyset)$ which is the *patched eigenvariety* constructed in [12]. Put $X_p(\bar{\rho}, \underline{\lambda}_J)' := X_p(\bar{\rho}) \times_{\widehat{T}_p} \widehat{T}_p(\underline{\lambda}_J)$, $X_p(\bar{\rho}, \underline{\lambda}_J, J') := X_p(\bar{\rho}, \underline{\lambda}_{J'}) \times_{\widehat{T}_p(\underline{\lambda}_{J'})} \widehat{T}_p(\underline{\lambda}_J)$, we have thus a commutative diagram (compare with (6)):

$$(19) \quad \begin{array}{ccccccccc} X_p(\bar{\rho}, \underline{\lambda}_J) & \longrightarrow & X_p(\bar{\rho}, \underline{\lambda}_J, J') & \longrightarrow & X_p(\bar{\rho}, \underline{\lambda}_{J'}) & \longrightarrow & X_p(\bar{\rho}, \underline{\lambda}_{J'})' & \longrightarrow & X_p(\bar{\rho}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widehat{T}_p(\underline{\lambda}_J) & \longrightarrow & \widehat{T}_p(\underline{\lambda}_J) & \longrightarrow & \widehat{T}_p(\underline{\lambda}_{J'}) & \longrightarrow & \widehat{T}_p(\underline{\lambda}_{J'}) & \longrightarrow & \widehat{T}_p \end{array}$$

where the horizontal maps are closed embeddings, and the second and fourth square are cartesian.

3.3.3. Structure of $X_p(\bar{\rho}, \underline{\lambda}_J)$. Keep the notation of the above section, and we fix an isomorphism $\mathcal{O}_E[\mathbb{Z}_p^q] \cong S_\infty$. Since Π_∞^\vee is a finite projective $S_\infty[\mathbb{K}_p][\frac{1}{p}]$ -module, so is $(\Pi_\infty \otimes_E L(\underline{\lambda}_J)')^\vee$. Thus for any prop- p compact open subgroup K'_p of K_p , one has

$$(\Pi_\infty \otimes_E L(\underline{\lambda}_J)')|_{\mathbb{Z}_p^q \times K'_p} \cong \mathcal{C}(\mathbb{Z}_p^q \times K'_p, E) \cong \mathcal{C}(\mathbb{Z}_p^q, E) \widehat{\otimes}_E \mathcal{C}(K'_p, E),$$

$$(20) \quad (\Pi_\infty^{R_\infty-\text{an}} \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J - \text{an}}|_{\mathbb{Z}_p^q \times K'_p} \cong \mathcal{C}^{\text{an}}(\mathbb{Z}_p^q, E) \widehat{\otimes}_E \mathcal{C}^{\Sigma_p \setminus J - \text{an}}(K'_p, E).$$

For $v|p$, let $z_{\bar{v}} := \begin{pmatrix} \varpi_{\bar{v}} & 0 \\ 0 & 1 \end{pmatrix} \in T_{\bar{v}}^-$, T_p^+ be the monoid in T_p generated by T_p^0 and $z_{\bar{v}}$ for all $v|p$, and $z := (z_{\bar{v}})_{v|p} \in T_p$, one has a natural projection $\widehat{T}_p \twoheadrightarrow \widehat{T}_p^0 \times \mathbb{G}_m$, $\delta \mapsto \delta|_{T_p^0} \times \delta(z)$ which induces projections $\widehat{T}_{p, \Sigma_p \setminus J} \twoheadrightarrow \widehat{T}_{p, \Sigma_p \setminus J}^0 \times \mathbb{G}_m$, $\widehat{T}_p(\underline{\lambda}_J) \twoheadrightarrow \widehat{T}_p^0(\underline{\lambda}_J) \times \mathbb{G}_m$. Denote by $\mathcal{W}_\infty := (\text{Spf } S_\infty)^{\text{rig}} \times \widehat{T}_p^0$, $\mathcal{W}_{\infty, \Sigma_p \setminus J} := (\text{Spf } S_\infty)^{\text{rig}} \times \widehat{T}_{p, \Sigma_p \setminus J}^0$, and $\mathcal{W}_\infty(\underline{\lambda}_J) := (\text{Spf } S_\infty)^{\text{rig}} \times \widehat{T}_p^0(\underline{\lambda}_J)$. The natural morphism $(\text{Spf } R_\infty)^{\text{rig}} \rightarrow (\text{Spf } S_\infty)^{\text{rig}}$ together with the above projection gives thus a morphism

$$\kappa : \mathfrak{X}_\infty \times \widehat{T}_p \longrightarrow \mathcal{W}_\infty \times \mathbb{G}_m.$$

Consider $J_{B_p}((\Pi_\infty^{R_\infty-\text{an}} \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J - \text{an}})$ which is a locally $\Sigma_p \setminus J$ -analytic representation of T_p equipped with a continuous action of R_∞ , and is essentially admissible as a $T_p \times \mathbb{Z}_p^q$ -representation. By (20) and (the proof of) [22, Prop.4.2.36], $J_{B_p}((\Pi_\infty^{R_\infty-\text{an}} \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J - \text{an}})^\vee$ is moreover a coadmissible $\mathcal{O}(\mathcal{W}_\infty \times \mathbb{G}_m)$ -module, and hence a coadmissible $\mathcal{O}(\mathcal{W}_{\infty, \Sigma_p \setminus J} \times \mathbb{G}_m)$ -module (since the action of $\mathcal{O}(\mathcal{W}_\infty)$ factors through $\mathcal{W}_{\infty, \Sigma_p \setminus J}$). By the isomorphism (18), $J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\lambda}_J))^\vee$ is thus a coadmissible $\mathcal{O}(\mathcal{W}_\infty(\underline{\lambda}_J) \times \mathbb{G}_m)$ -module, in other words, the push forward $\mathcal{N}(\bar{\rho}, \underline{\lambda}_J) := \kappa_* \mathcal{M}(\bar{\rho}, \underline{\lambda}_J)$ is a coherent sheaf on $\mathcal{W}_\infty(\underline{\lambda}_J) \times \mathbb{G}_m$.

Lemma 3.9. (1) There exist an admissible covering of $\mathcal{W}_{\infty, \Sigma_p \setminus J}$ by affinoid opens $U_1 \subset U_2 \subset \dots \subset U_h \subset \dots$, and a commutative diagram

$$\begin{array}{ccccccc} \left(\left((\Pi_{\infty}^{R_{\infty}-\text{an}} \otimes_E L(\underline{\Delta}_J)')^{\Sigma_p \setminus J-\text{an}} \right)^{N_p^0} \right)^{\vee} & \longrightarrow & \dots & \longrightarrow & V_{h+1} & \longrightarrow & V_{h+1} \otimes_{A_{h+1}} A_h \xrightarrow{\beta_h} V_n \\ \downarrow z & & & & \downarrow z_{h+1} & & \swarrow \alpha_h \searrow \downarrow z_h \\ \left(\left((\Pi_{\infty}^{R_{\infty}-\text{an}} \otimes_E L(\underline{\Delta}_J)')^{\Sigma_p \setminus J-\text{an}} \right)^{N_p^0} \right)^{\vee} & \longrightarrow & \dots & \longrightarrow & V_{h+1} & \longrightarrow & V_{h+1} \otimes_{A_{h+1}} A_h \xrightarrow{\beta_h} V_h \end{array}$$

where $A_h = \Gamma(U_h, \mathcal{O}_{\mathcal{W}_{\infty, \Sigma_p \setminus J}})$, V_h is a Banach A_h -module equipped with a compact A_h -linear operator z_h , α_h, β_h are A_h -linear, and β_h is compact, $\beta_h \circ \alpha_h = z_h$, $\alpha_h \circ \beta_h = z_{h+1} \otimes 1_{A_h}$, and one has an isomorphism of $\mathcal{O}(\mathcal{W}_{\infty, \Sigma_p \setminus J})$ -modules

$$\left(\left((\Pi_{\infty}^{R_{\infty}-\text{an}} \otimes_E L(\underline{\Delta}_J)')^{\Sigma_p \setminus J-\text{an}} \right)^{N_p^0} \right)^{\vee} \xrightarrow{\sim} \varprojlim_h V_h$$

which commutes with the Hecke action of z on the left object, and that of $(z_h)_{h \in \mathbb{Z}_{\geq 1}}$ on the right.

(2) The statement in (1) also holds with the rigid space $\mathcal{W}_{\infty, \Sigma_p \setminus J}$ replaced by $\mathcal{W}_{\infty}(\underline{\Delta}_J)$, and $\left((\Pi_{\infty}^{R_{\infty}-\text{an}} \otimes_E L(\underline{\Delta}_J)')^{\Sigma_p \setminus J-\text{an}} \right)^{N_p^0}$ replaced by $(\Pi_{\infty}^{R_{\infty}-\text{an}}(\underline{\Delta}_J))^{N_p^0}$.

Proof. (1) follows from the fact (20) as in [12, Prop.5.3] (see also [22, Prop.4.2.36]).

(2) is follows from (1) together with the isomorphism

$$(\Pi_{\infty}^{R_{\infty}-\text{an}}(\underline{\Delta}_J))^{N_p^0} \xrightarrow{\sim} \left((\Pi_{\infty}^{R_{\infty}-\text{an}} \otimes_E L(\underline{\Delta}_J)')^{\Sigma_p \setminus J-\text{an}} \right)^{N_p^0} \otimes_E \delta_{\underline{\Delta}_J},$$

which is $R_{\infty} \times T_p^+$ -invariant. \square

Denote by $Z(\overline{\rho}, \underline{\Delta}_J)$ the support of $\mathcal{N}(\overline{\rho}, \underline{\Delta}_J)$ in $\mathcal{W}_{\infty}(\underline{\Delta}_J) \times \mathbb{G}_m$. The morphism κ induces thus a morphism $\kappa : X(\overline{\rho}, \underline{\Delta}_J) \rightarrow Z(\overline{\rho}, \underline{\Delta}_J)$. Denote by $g : Z(\overline{\rho}, \underline{\Delta}_J) \rightarrow \mathcal{W}_{\infty}(\underline{\Delta}_J)$ and $\omega_{\infty} : X_p(\overline{\rho}, \underline{\Delta}_J) \rightarrow \mathcal{W}_{\infty}(\underline{\Delta}_J)$ the natural morphisms, thus $\omega_{\infty} = g \circ \kappa$. As in [12, Lem.2.10, Prop.3.11], one deduces from the above lemma:

Proposition 3.10. (1) $Z(\overline{\rho}, \underline{\Delta}_J)$ is a Fredholm hypersurface in $\mathcal{W}_{\infty}(\underline{\Delta}_J) \times \mathbb{G}_m$, and there exists an admissible covering $\{U_i\}_{i \in I}$ of $Z(\overline{\rho}, \underline{\Delta}_J)$ by affinoids U_i such that the morphism g induces a finite surjective map from U_i to an affinoid open W_i of $\mathcal{W}_{\infty}(\underline{\Delta}_J)$ and U_i is a connected component of $g^{-1}(W_i)$. Moreover, for $i \in I$, $\Gamma(U_i, \mathcal{N}_{\infty}(\underline{\Delta}_J))$ is a finite projective $\Gamma(W_i, \mathcal{O}_{\mathcal{W}_{\infty}(\underline{\Delta}_J)})$ -module.

(2) There exists an admissible covering $\{\mathcal{U}_i\}_{i \in I}$ of $X_p(\overline{\rho}, \underline{\Delta}_J)$ by affinoids \mathcal{U}_i such that for all i there exists an open affinoid W_i of $\mathcal{W}_{\infty}(\underline{\Delta}_J)$ such that the morphism ω_{∞} induces a finite surjective morphism from each irreducible component of \mathcal{U}_i onto W_i and that $\Gamma(\mathcal{U}_i, \mathcal{O}_{X_p(\overline{\rho})})$ is isomorphic to a $\Gamma(W_i, \mathcal{O}_{\mathcal{W}_{\infty}(\underline{\Delta}_J)})$ -subalgebra of the endomorphism ring of a finite projective $\Gamma(W_i, \mathcal{O}_{\mathcal{W}_{\infty}(\underline{\Delta}_J)})$ -module.

Remark 3.11. As in the proof of [12, Prop.3.11], one can take \mathcal{U}_i (in (2)) to be $\kappa^{-1}(U_i)$ (with U_i as in (1)), and then $\Gamma(\mathcal{U}_i, \mathcal{O}_{X_p(\overline{\rho})})$ is just the $\Gamma(W_i, \mathcal{O}_{\mathcal{W}_{\infty}(\underline{\Delta}_J)})$ -subalgebra of the endomorphism ring of the finite projective $\Gamma(W_i, \mathcal{O}_{\mathcal{W}_{\infty}(\underline{\Delta}_J)})$ -module $\Gamma(U_i, \mathcal{N}_{\infty}(\underline{\Delta}_J)) \cong \Gamma(\mathcal{U}_i, \mathcal{M}_{\infty}(\underline{\Delta}_J))$ generated by the operators in $R_{\infty} \times T_p$.

Corollary 3.12. (1) The rigid space $X_p(\overline{\rho}, \underline{\Delta}_J)$ is equidimensional of dimension

$$g + 4|S| + 3[F^+ : \mathbb{Q}] - 2|J|,$$

locally finite over $\mathcal{W}_{\infty}(\underline{\Delta}_J)$ and does not have any embedded component.

(2) The coherent sheaf $\mathcal{M}_{\infty}(\underline{\Delta}_J)$ is Cohen-Macaulay over $X_p(\overline{\rho}, \underline{\Delta}_J)$.

Proof. (1) follows by the same argument as in [12, Cor.3.12] (see also [15, Prop.6.4.2]).

(2) follows by the same argument as in [13, Lem.3.8]. \square

With the notation of (19), one has (by definitions and Cor.3.12):

Corollary 3.13. (1) $\dim X_p(\bar{\rho}, \underline{\Delta}_{J'})' = g + 4|S| + 3[F^+ : \mathbb{Q}] - 2|J'|$ (of the same dimension of $X_p(\bar{\rho}, \underline{\Delta}_{J'})$);
(2) $\dim X_p(\bar{\rho}, \underline{\Delta}_J, J') = g + |4|S| + 3[F^+ : \mathbb{Q}] - 2|J|$ (of the same dimension of $X_p(\bar{\rho}, \underline{\Delta}_J)$).

3.3.4. *Density of classical points.* Recall a point (y, δ) of $X_p(\bar{\rho})$ is called *classical* if

$$J_{B_p}(\Pi_{\infty}^{R_{\infty}-\text{an}, \text{lalg}})[\mathfrak{p}_y, T_p = \delta] \neq 0,$$

where “lalg” denotes the locally algebraic vectors for the G_p -action; and a point (y, δ) of $X_p(\bar{\rho}, \underline{\Delta}_J)$ is called classical if it’s classical as a point in $X_p(\bar{\rho})$, which is also equivalent to

$$J_{B_p}(\Pi_{\infty}^{R_{\infty}-\text{an}}(\underline{\Delta}_J)^{\text{lalg}})[\mathfrak{p}_y, T_p = \delta] \neq 0.$$

A closed point (y, δ) of $X_p(\bar{\rho})$ is called *spherical* if δ is locally algebraic (i.e. $\text{wt}(\delta) \in \mathbb{Z}^{2|\Sigma_p|}$) and $\psi_{\delta} := \delta \delta_{\text{wt}(\delta)}^{-1}$ is unramified; (y, δ) is called *very regular* if δ is locally algebraic and the character

$$(21) \quad \delta_v^{\natural} := \delta_{\bar{v}}(\text{unr}_{\bar{v}}(q_{\bar{v}}) \otimes \prod_{\sigma \in \Sigma_{\bar{v}}} \sigma^{-1})$$

is very regular (cf. (3)) for all $v|p$. Let $\delta^{\natural} := \prod_{v|p} \delta_v^{\natural}$.

For a locally algebraic character δ of T_p , let

$$(22) \quad N(\delta) := \{\sigma \in \Sigma_p \mid \text{wt}(\delta)_{1,\sigma} \geq \text{wt}(\delta)_{2,\sigma}\},$$

and for $J' \subseteq N(\delta)$, put

$$(23) \quad \delta_{J'}^c := \otimes_{v|p} \delta_{\bar{v}, J'_v}^c := \otimes_{v|p} (\delta_{\bar{v}}(\prod_{\sigma \in J'_v} \sigma^{\text{wt}(\delta)_{2,\sigma} - \text{wt}(\delta)_{1,\sigma} - 1} \otimes \prod_{\sigma \in J'_v} \sigma^{\text{wt}(\delta)_{1,\sigma} - \text{wt}(\delta)_{2,\sigma} + 1})).$$

Thus $\text{wt}(\delta_{J'}^c)_{i,\sigma} = \text{wt}(\delta)_{i,\sigma}$ if $\sigma \notin J'$; $\text{wt}(\delta_{J'}^c)_{1,\sigma} = \text{wt}(\delta)_{2,\sigma} - 1$, $\text{wt}(\delta_{J'}^c)_{2,\sigma} = \text{wt}(\delta)_{1,\sigma} + 1$ if $\sigma \in J'$, and $N(\delta_{J'}^c) = N(\delta) \setminus J'$. In fact, let $s_{J'} := \prod_{\sigma \in J'} s_{\sigma}$ where s_{σ} denotes the (unique) simple reflection in the Weyl group S_2 of $\mathfrak{g}_{\bar{v}} \otimes_{F_{\bar{v}}, \sigma} E$ (note the Weyl group of $\mathfrak{g}_p \otimes_{\mathbb{Q}_p} E$ is the product of the Weyl groups S_2 of all $\mathfrak{g}_{\bar{v}} \otimes_{F_{\bar{v}}, \sigma} E$), then $\delta_{J'}^c = \psi_{\delta} \delta_{s_{J'} \cdot \text{wt}(\delta)}$ (where “ \cdot ” denotes the dot action).

For a locally \mathbb{Q}_p -analytic character δ of T_p over E , put

$$(24) \quad I(\delta) := (\text{Ind}_{\bar{B}_p}^{G_p} \delta \delta_{\text{wt}(\delta)_{N(\delta)}}^{-1})^{\Sigma_p \setminus N(\delta) - \text{an}} \otimes_E L(\text{wt}(\delta)_{N(\delta)}),$$

by [8, Thm.4.1] (see also Prop.B.4 below), the representations $\{I(\delta_{J'}^c)\}_{J' \subseteq N(\delta)}$ give the Jordan-Holder components of $(\text{Ind}_{\bar{B}_p}^{G_p} \delta)^{\mathbb{Q}_p - \text{an}}$ (and also of $\mathcal{F}_{\bar{B}_p}^{G_p}(M(-\text{wt}(\delta))^{\vee}, \psi_{\delta})$, see App.B for notations). Note also that $I(\delta)$ is locally algebraic if δ is dominant. For $v|p$, put

$$I_{\bar{v}}(\delta_{\bar{v}}) := (\text{Ind}_{\bar{B}(F_{\bar{v}})}^{\text{GL}_2(F_{\bar{v}})} (\delta \delta_{\text{wt}(\delta)_{N(\delta)}}^{-1})_{\bar{v}})^{\Sigma_{\bar{v}} \setminus N(\delta)_{\bar{v}} - \text{an}} \otimes_E L(\text{wt}(\delta)_{N(\delta)_{\bar{v}}}),$$

where $N(\delta)_{\bar{v}} := N(\delta) \cap \Sigma_{\bar{v}}$, which is a locally \mathbb{Q}_p -analytic representation of $\text{GL}_2(F_{\bar{v}})$, and we have

$$(25) \quad I(\delta) \xrightarrow{\sim} \otimes_{v|p} I_{\bar{v}}(\delta_{\bar{v}}).$$

Note also that $I_{\bar{v}}(\delta_{\bar{v}})$ is irreducible if $N(\delta)_{\bar{v}} \neq \Sigma_{\bar{v}}$. Denote by $\delta_{B_p} = \otimes_{v|p} (\text{unr}_{\bar{v}}(q_{\bar{v}}^{-1}) \otimes \text{unr}_{\bar{v}}(q_{\bar{v}}))$ the modulus character of B_p , which factors through T_p and thus can also be viewed as a character of \bar{B}_p via $\bar{B}_p \twoheadrightarrow T_p$.

Lemma 3.14. *Let $x = (y, \delta)$ be a closed point of $X_p(\overline{\rho}, \underline{\lambda}_J)$ (hence $\text{wt}(\delta)_J = \underline{\lambda}_J$ for $\sigma \in J$) with $\text{wt}(\delta)$ dominant, if any irreducible component of $I(\delta_{J'}^c, \delta_{B_p}^{-1})$ does not have G_p -invariant lattice for all $\emptyset \neq J' \subseteq \Sigma_p \setminus J$, then x is classical. We call such classical points $\Sigma_p \setminus J$ -very classical.*

Proof. By the adjunction formula Prop.B.5, one has

$$(26) \quad \text{Hom}_{T_p} \left(\delta, J_{B_p}(\Pi_{\infty}^{R_{\infty}-\text{an}}(\underline{\lambda}_J))[\mathfrak{p}_y] \right) \xrightarrow{\sim} \text{Hom}_{G_p} \left(\mathcal{F}_{B_p}^{G_p}(M_J(-\text{wt}(\delta))^{\vee}, \psi_{\delta} \delta_{B_p}^{-1}), \Pi_{\infty}^{R_{\infty}-\text{an}}(\underline{\lambda}_J)[\mathfrak{p}_y] \right).$$

Note the Jordan-Holder factors of $\mathcal{F}_{B_p}^{G_p}(M_J(-\text{wt}(\delta))^{\vee}, \psi_{\delta} \delta_{B_p}^{-1})$ are given by $\{I(\delta_{J'}^c, \delta_{B_p}^{-1})\}_{J' \subseteq \Sigma_p \setminus J}$. By assumption and that Π_{∞} is unitary, any non-zero map in the left set of (26) factors through $I(\delta_{B_p}^{-1})$, hence x is classical. \square

Proposition 3.15. *Let (y, δ) be a closed point of $X_p(\overline{\rho}, \underline{\lambda}_J)$ with $\text{wt}(\delta)$ dominant, if*

$$(27) \quad v_{\tilde{v}}(q_{\tilde{v}} \delta_{\tilde{v},1}(\varpi_{\tilde{v}})) < \inf_{\sigma \in \Sigma_{\tilde{v}} \setminus J_{\tilde{v}}} \{\text{wt}(\delta)_{1,\sigma} - \text{wt}(\delta)_{2,\sigma} + 1\}$$

for all $v|p$ with $J_{\tilde{v}} \neq \Sigma_{\tilde{v}}$, then the point (y, δ) is $\Sigma_p \setminus J$ -very classical.

Proof. Let $J' \subset \Sigma_p \setminus J$, $J' \neq \emptyset$, if there exists an irreducible component of $I(\delta_{J'}^c, \delta_{B_p}^{-1})$ which admits a G_p -invariant lattice, then by (25) there exists $v|p$ such that $J'_{\tilde{v}} \neq \emptyset$, and $I_{\tilde{v}}((\delta_{J'}^c)_{\tilde{v}} \delta_{B(F_{\tilde{v}})}^{-1})$ (which is irreducible since $J'_{\tilde{v}} \neq \emptyset$) admits a $\text{GL}_2(F_{\tilde{v}})$ -invariant lattice. By [8, Prop.5.1], we can deduce

$$v_{\tilde{v}}(q_{\tilde{v}} \delta_{\tilde{v},1}(\varpi_{\tilde{v}})) \geq \sum_{\sigma \in J'_{\tilde{v}}} (\text{wt}(\delta)_{1,\sigma} - \text{wt}(\delta)_{2,\sigma} + 1),$$

which contradicts to the assumption. The Proposition follows then by Lem.3.14. \square

The following theorem follows from Prop.3.15 by standard arguments as in [15].

Theorem 3.16. (1) *The set of classical points is Zariski-dense in $X_p(\overline{\rho}, \underline{\lambda}_J)$ and is an accumulate set in $X_p(\overline{\rho}, \underline{\lambda}_J)$.*

(2) *The set of spherical, very regular and $\Sigma_p \setminus J$ -very classical points accumulates over spherical very regular points in $X_p(\overline{\rho}, \underline{\lambda}_J)$.*

Proof. (1) For the first part, it's sufficient to prove the classical points are Zariski-dense in any irreducible component X of $X_p(\overline{\rho}, \underline{\lambda}_J)$. By [15, Cor.6.4.4] (see also Cor.3.12 (1)) and the fact that the locally algebraic characters of T_p^0 are Zariski-dense in $\widehat{T}_p^0(\underline{\lambda}_J)$, there exists $x = (y, \delta) \in X$ with δ locally algebraic. Let U be an irreducible affinoid open neighborhood of x in X such that $\omega_{\infty} : U \rightarrow \omega_{\infty}(U)$ is finite and $\omega_{\infty}(U) = V_1 \times V_2$ is an irreducible affinoid open in $\mathcal{W}_{\infty}(\underline{\lambda}_J) = (\text{Spf } S_{\infty})^{\text{rig}} \times \widehat{T}_p^0(\underline{\lambda}_J)$ (shrinking U if necessary). Let

$$C_1 := \max_{(y', \delta') \in U, v|p, J_{\tilde{v}} \neq \Sigma_{\tilde{v}}} v_{\tilde{v}}(q_{\tilde{v}} \delta'_{1,\tilde{v}}(\varpi_{\tilde{v}})) \in \mathbb{Q}.$$

Let $C_2 \geq C_1$, the set Z of points δ' in V_2 , satisfying

- (i) $\text{wt}(\delta')$ is integral,
- (ii) $\inf_{\sigma \in J_{\tilde{v}}} \{\text{wt}(\delta')_{1,\sigma} - \text{wt}(\delta')_{2,\sigma} + 1\} \geq C_2$ for $v|p$ with $J_{\tilde{v}} \neq \Sigma_{\tilde{v}}$,

is Zariski dense in V_2 . By [15, Lem.6.2.8], $\omega_{\infty}^{-1}(V_1 \times Z)$ is Zariski-dense in U . However, by Prop.3.15, any point in $\omega_{\infty}^{-1}(V_1 \times Z)$ is $\Sigma_p \setminus J$ -very classical. The first part of (1) follows. In fact, the above argument shows the classical points accumulate over x (since the set Z in fact accumulates over δ), thus the second part of (1) also follows.

(2) If x (with the above notation) is moreover spherical very regular, we can choose Z such that all the characters in Z are *algebraic*, thus any point in $\omega_\infty^{-1}(V_1 \times Z)$ is spherical and $\Sigma_p \setminus J$ -very classical.

If $J_{\tilde{v}} = \Sigma_{\tilde{v}}$, then for any (y', δ') in U , $\delta'|_{T_{\tilde{v}}^0} = \delta|_{T_{\tilde{v}}^0} = \delta_{\Delta_{\Sigma_{\tilde{v}}}}|_{T_{\tilde{v}}^0}$ (since $V_1 \times V_2$ is irreducible, and $\widehat{T}_{\tilde{v}}(\Delta_{\Sigma_{\tilde{v}}})$ is discrete). We shrink U such that $(\delta'_{\tilde{v}})^{\natural}$ is very regular (cf. (21) (3), note in this case this is equivalent to $(\delta'_{\tilde{v}})^{\natural}$ being regular) for all $v|p$ with $J_{\tilde{v}} = \Sigma_{\tilde{v}}$ and all (y', δ') in U .

Note that for any point (y', δ') in $X_p(\overline{\rho}, \underline{\lambda}_J)$, one has

$$v_{\tilde{v}}(\delta'_{1,\tilde{v}}(\varpi_{\tilde{v}})) + v_{\tilde{v}}(\delta'_{2,\tilde{v}}(\varpi_{\tilde{v}})) = 0$$

for all $v|p$ (since Π_∞ is unitary in particular for the action of the center). Thus for $v|p$, $J_{\tilde{v}} \neq \Sigma_{\tilde{v}}$, if C_2 is big enough, the property (ii) would imply that $(\delta'_{\tilde{v}})^{\natural}$ is very regular.

Thus taking C_2 big enough together with the above assumption for U , any point in $\omega_\infty^{-1}(V_1 \times Z)$ would be moreover very regular. Part (2) follows. \square

Remark 3.17. If $J_{\tilde{v}} \neq \Sigma_{\tilde{v}}$ for all $v|p$, one can in fact prove (by the same arguments as above) that the spherical, very regular, $\Sigma_p \setminus J$ -very classical points are Zariski dense in $X_p(\overline{\rho}, \underline{\lambda}_J)$. However, if $J_{\tilde{v}} = \Sigma_{\tilde{v}}$, let $x = (y, \delta) \in X_p(\overline{\rho}, \underline{\lambda}_J)$, thus $\text{wt}(\delta)_\sigma = (\lambda_{1,\sigma}, \lambda_{2,\sigma})$ for $\sigma \in J$ (thus for $\sigma \in \Sigma_{\tilde{v}}$); since $\widehat{T}_{\tilde{v}}(\Delta_{\Sigma_{\tilde{v}}})$ consists of isolated points, if $x' = (r', \delta')$ lies in an irreducible component of $X_p(\overline{\rho}, \underline{\lambda}_J)$ containing x , then $\delta'|_{T_{\tilde{v}}^0} = \delta|_{T_{\tilde{v}}^0}$. In particular, if $\delta_{\tilde{v}} \delta_{\Delta_{\tilde{v}}}^{-1}$ is not unramified, then an irreducible component containing x does not have spherical classical points.

3.3.5. Reducedness of patched eigenvarieties.

Theorem 3.18. $X_p(\overline{\rho}, \underline{\lambda}_J)$ is reduced at spherical very regular points.

Proof. This theorem follows from the same argument as in the proof of [12, Cor.3.20]. Let $x = (y, \delta)$ be a spherically classical point, by Prop.3.10 (see also Rem.3.11), there exists an affinoid open neighborhood $U = \text{Spm } B$ of x in $X_p(\overline{\rho}, \underline{\lambda}_J)$ such that

- $\omega_\infty(U) = \text{Spm } A$ is an irreducible affinoid open in $(\text{Spf } S_\infty)^{\text{rig}} \times \widehat{T}_p^0(\underline{\lambda}_J)$,
- $\omega_\infty(U)$ has the form $V_1 \times V_2 \subseteq (\text{Spf } S_\infty)^{\text{rig}} \times \widehat{T}_p^0(\underline{\lambda}_J)$ (with V_i irreducible),
- $M := \Gamma(U, \mathcal{M}_\infty(\underline{\lambda}_J))$ is a finite projective A -module equipped with a B -linear action of $R_\infty \times T_p^+$, and B is isomorphic to the B -subalgebra of $\text{End}_A(M)$ generated by R_∞ and T_p^+ .

Note if $J_{\tilde{v}} = \Sigma_{\tilde{v}}$, then image of U in $\widehat{T}_{\tilde{v}}^0$ is a single point. As in the proof of Thm.3.16 (2), we can find a set $Z \subset V_2(\overline{E})$ such that (shrinking U if necessary)

- (1) Z is Zariski-dense in V_2 ,
- (2) any point $x' \in \omega_\infty^{-1}(V_1 \times Z)$ is spherical, very regular and $\Sigma_p \setminus J$ -very classical.

As in the proof of [16, Prop.3.9] (see also the proof of [12, Cor.3.20]), it's sufficient to prove for any $\mu \in Z$ (which we can and do view as a weight of \mathfrak{t}_p), there exists an affinoid open U_μ of V_1 (which is thus Zariski-dense in V_1) such that for any $z \in U_\mu \times \{\mu\}$, the action of B (thus of R_∞ and T_p^+) on $M \otimes_A k(z)$ is semi-simple, where $k(z)$ denotes the residue field at z .

Let $z \in V_1 \times \{\mu\}$, \mathfrak{p}_z the associated maximal ideal of $S_\infty[\frac{1}{p}]$, $\Sigma := \text{Hom}_{k(z)}(M \otimes_A k(z), k(z))$ (of finite dimension over $k(z)$), which is isomorphic to a direct factor of $J_{B_p}((\Pi_\infty^{R_\infty - \text{an}}(\underline{\lambda}_J))[T_p^0 = \mu])$ and thus has the form $\Sigma \cong \delta_\mu \otimes_E \Sigma_\infty$ where δ_μ denotes the algebraic character of weight μ and Σ_∞ is a finite

dimensional smooth unramified representation of T_p . By Prop.B.5, one has

$$\begin{aligned} \mathrm{Hom}_{T_p} \left(\delta_\mu \otimes_E \Sigma_\infty, J_{B_p} \left(\Pi_\infty^{R_\infty - \mathrm{an}}(\underline{\lambda}_J)[\mathfrak{p}_z] \right) \right) \\ \xrightarrow{\sim} \mathrm{Hom}_{G_p} \left(\mathcal{F}_{B_p}^{G_p} \left(M_J(-\mu)^\vee, \Sigma_\infty \otimes_{k(z)} \delta_{B_p}^{-1} \right), \Pi_\infty^{R_\infty - \mathrm{an}}(\underline{\lambda}_J)[\mathfrak{p}_z] \right). \end{aligned}$$

Since any point in $\omega_\infty^{-1}(z)$ is $\Sigma_p \setminus J$ -very classical, one can show as in the proof of Lem.3.14 that any morphism on the right side factors through the locally algebraic quotient

$$\mathcal{F}_{B_p}^{G_p} \left(L(-\mu), \Sigma_\infty \otimes_{k(z)} \delta_{B_p}^{-1} \right) \cong \left(\mathrm{Ind}_{B_p}^{G_p} \Sigma_\infty \otimes_{k(z)} \delta_{B_p}^{-1} \right)^\infty \otimes_E L(\mu).$$

The theorem then follows from verbatim of the last paragraph of the proof of [12, Cor.3.20]. \square

Since $X_p(\bar{\rho}, \underline{\lambda}_J)$ does not have embedded component, by the arguments in the first paragraph of the proof of [12, Cor.3.20], one has

Corollary 3.19. *Let X be a union of irreducible components of $X_p(\bar{\rho}, \underline{\lambda}_J)$ with each irreducible components containing a spherical very regular point, then X is reduced.*

Remark 3.20. *In particular, as discussed in Rem.3.17, if $J_{\tilde{v}} \neq \Sigma_{\tilde{v}}$ for all $v|p$, then any irreducible component of $X_p(\bar{\rho}, \underline{\lambda}_J)$ contains spherical very regular points, and hence $X_p(\bar{\rho}, \underline{\lambda}_J)$ is reduced.*

3.4. Infinitesimal “R=T” results. Let $\mathfrak{X}_{\bar{\rho}^p}^\square := \mathrm{Spf} \left(\widehat{\otimes}_{v \in S \setminus \Sigma_p} R_{\bar{\rho}_v}^\square \right)^{\mathrm{rig}}$, $\mathfrak{X}_{\bar{\rho}_p}^\square := \mathrm{Spf} \left(\widehat{\otimes}_{v|p} R_{\bar{\rho}_v}^\square \right)^{\mathrm{rig}}$, and \mathbb{U} be the open unit ball in \mathbb{A}^1 , we have thus $(\mathrm{Spf} R_\infty)^{\mathrm{rig}} \cong \mathfrak{X}_{\bar{\rho}^p}^\square \times \mathbb{U}^g \times \mathfrak{X}_{\bar{\rho}_p}^\square$. For $v|p$, denote by $\iota_{\tilde{v}}$ the following isomorphism (cf. (21))

$$\iota_{\tilde{v}} : \widehat{T}_{\tilde{v}} \xrightarrow{\sim} \widehat{T}_{\tilde{v}}, \quad \delta_{\tilde{v}} \mapsto \delta_{\tilde{v}}^{\natural},$$

put $\iota_p := \prod_{v|p} \iota_{\tilde{v}} : \widehat{T}_p \xrightarrow{\sim} \widehat{T}_p$, $\iota_{\tilde{v}}^{-1}(X_{\mathrm{tri}}^\square(\bar{\rho}_{\tilde{v}})) := X_{\mathrm{tri}}^\square(\bar{\rho}_{\tilde{v}}) \times_{\widehat{T}_{\tilde{v}}, \iota_{\tilde{v}}} \widehat{T}_{\tilde{v}}$, $X_{\mathrm{tri}}^\square(\bar{\rho}_p) := \prod_{v|p} X_{\mathrm{tri}}^\square(\bar{\rho}_{\tilde{v}})$, and

$$\iota_p^{-1}(X_{\mathrm{tri}}^\square(\bar{\rho}_p)) := X_{\mathrm{tri}}^\square(\bar{\rho}_p) \times_{\widehat{T}_p, \iota_p} \widehat{T}_p \cong \prod_{v|p} \iota_{\tilde{v}}^{-1}(X_{\mathrm{tri}}^\square(\bar{\rho}_{\tilde{v}})).$$

Note $\iota_p^{-1}(X_{\mathrm{tri}}^\square(\bar{\rho}_p))$ is a closed subspace of $\mathfrak{X}_{\bar{\rho}^p}^\square \times \widehat{T}_p$. Recall

Theorem 3.21 ([12, Thm.3.21]). *The natural embedding $X_p(\bar{\rho}) \hookrightarrow (\mathrm{Spf} R_\infty)^{\mathrm{rig}} \times \widehat{T}_p$ factors through*

$$(28) \quad X_p(\bar{\rho}) \hookrightarrow \mathfrak{X}_{\bar{\rho}^p}^\square \times \mathbb{U}^g \times \iota_p^{-1}(X_{\mathrm{tri}}^\square(\bar{\rho}_p)),$$

and induces an isomorphism between $X_p(\bar{\rho})$ and a union of irreducible components (equipped with the reduced closed rigid subspace structure) of $\mathfrak{X}_{\bar{\rho}^p}^\square \times \mathbb{U}^g \times \iota_p^{-1}(X_{\mathrm{tri}}^\square(\bar{\rho}_p))$.

For a closed point $x = (y, \delta)$ of $X_p(\bar{\rho})$, and $v \in S$ denote by $\rho_{x, \tilde{v}}$ its image in $(\mathrm{Spf} R_{\bar{\rho}_v}^\square)^{\mathrm{rig}}$ via the natural morphism; for $v|p$, $x_{\tilde{v}} = (\rho_{x, \tilde{v}}, \delta_{\tilde{v}}^{\natural})$ is thus the image of x in $X_{\mathrm{tri}}^\square(\bar{\rho}_{\tilde{v}})$ via (28). For a finite place \tilde{v} of F , $v \nmid p$, and a 2-dimensional p -adic representation $\rho_{\tilde{v}}$ of $\mathrm{Gal}_{F_{\tilde{v}}}$, $\rho_{\tilde{v}}$ is called *generic* if the smooth representation π of $\mathrm{GL}_2(F_{\tilde{v}})$ associated to $\mathrm{WD}(\rho_{\tilde{v}})$ (the associated Weil-Deligne representation) via local Langlands correspondance is generic (which is equivalent to infinite dimensional in this case).

For a weight $\underline{\lambda}_{\Sigma_p} = (\lambda_{1, \sigma}, \lambda_{2, \sigma})_{\sigma \in \Sigma_p}$ of $\mathfrak{t}_p \otimes_{\mathbb{Q}_p} E$, $S \subseteq \Sigma_p$, denote by $\underline{\lambda}_S^{\natural} := (\lambda_{1, \sigma}, \lambda_{2, \sigma} - 1)_{\sigma \in S}$. Now let $J \subseteq \Sigma_p$, $\underline{\lambda}_J \in \mathbb{Z}^{2|J|}$ be dominant, put $X_{\mathrm{tri}}^\square(\bar{\rho}_p, \underline{\lambda}_J^{\natural}) := \prod_{v|p} X_{\mathrm{tri}}^\square(\bar{\rho}_{\tilde{v}}, \underline{\lambda}_{J_{\tilde{v}}}^{\natural})$ (cf. §2.1). By definitions, the closed embedding (28) induces a closed embedding

$$(29) \quad X_p(\bar{\rho}, \underline{\lambda}_J)' \hookrightarrow \mathfrak{X}_{\bar{\rho}^p}^\square \times \mathbb{U}^g \times \iota_p^{-1}(X_{\mathrm{tri}}^\square(\bar{\rho}_p, \underline{\lambda}_J^{\natural}))$$

which is moreover a local isomorphism by Thm.3.21. Put $X_{\text{tri}, J-\text{dR}}^{\square}(\bar{\rho}_p, \underline{\Delta}_J^{\natural}) := \prod_{v|p} X_{\text{tri}, J_v-\text{dR}}^{\square}(\bar{\rho}_v, \underline{\Delta}_{J_v}^{\natural})$ (cf. (4)). By Shah's results [31] and the density of classical points in $X_p(\bar{\rho}, \underline{\Delta}_J)$, the injection (28) induces a closed embedding

$$(30) \quad X_p(\bar{\rho}, \underline{\Delta}_J)_{\text{red}} \hookrightarrow \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g \times \iota_p^{-1}(X_{\text{tri}, J-\text{dR}}^{\square}(\bar{\rho}_p, \underline{\Delta}_J^{\natural})).$$

Theorem 3.22. *Let x be a spherical, very regular point of $X_p(\bar{\rho}, \underline{\Delta}_J)$ (thus $X_p(\bar{\rho}, \underline{\Delta}_J)$ is reduced at x by Thm.3.18), suppose $\rho_{x, \tilde{v}}$ is generic for $v \in S \setminus S_p$, then $X_p(\bar{\rho}, \underline{\Delta}_J)$ is smooth at x , $X_{\text{tri}, J_v-\text{dR}}^{\square}(\bar{\rho}_v, \underline{\Delta}_{J_v}^{\natural})$ is smooth at x_v for $v|p$, and the closed embedding (30) induces an isomorphism of complete regular noetherian local $k(x)$ -algebras:*

$$(31) \quad \widehat{\mathcal{O}}_{X_p(\bar{\rho}, \underline{\Delta}_J), x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g \times \iota_p^{-1}(X_{\text{tri}, J-\text{dR}}^{\square}(\bar{\rho}_p, \underline{\Delta}_J^{\natural})), x}.$$

Proof. Put $Z := \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g \times \iota_p^{-1}(X_{\text{tri}, J-\text{dR}}^{\square}(\bar{\rho}_p, \underline{\Delta}_J^{\natural}))$ for simplicity. Since $X_p(\bar{\rho}, \underline{\Delta}_J)$ is reduced at x , the closed embedding (30) induces a surjective morphism

$$\widehat{\mathcal{O}}_{Z, x} \twoheadrightarrow \widehat{\mathcal{O}}_{X_p(\bar{\rho}, \underline{\Delta}_J), x},$$

thus $\dim \widehat{\mathcal{O}}_{Z, x} \geq \dim \widehat{\mathcal{O}}_{X_p(\bar{\rho}, \underline{\Delta}_J), x} = g + 4|S| + 3[F^+ : \mathbb{Q}] - 2|J|$; on the other hand, since $\rho_{x, \tilde{v}}$ is generic for $v \in S \setminus S_p$, by [1, Lem.1.3.2 (1)], $\mathfrak{X}_{\bar{\rho}^p}$ is smooth at $(\rho_{x, \tilde{v}})_{v \in S \setminus S_p}$ (of dimension $4|S \setminus S_p|$), which together with Thm.2.5 (1) calculates:

$$\dim_{k(x)} T_{Z, x} = g + 4|S| + 3[F^+ : \mathbb{Q}] - 2|J|.$$

The theorem follows. \square

Recall $X_p(\bar{\rho}, \underline{\Delta}_J, J') \cong X_p(\bar{\rho}, \underline{\Delta}_{J'}) \times_{\widehat{T}(\underline{\Delta}_{J'})} \widehat{T}(\underline{\Delta}_J)$; put (cf. (5))

$$X_{\text{tri}, J'-\text{dR}}^{\square}(\bar{\rho}_p, \underline{\Delta}_J^{\natural}) := \prod_{v|p} X_{\text{tri}, J'_v-\text{dR}}^{\square}(\bar{\rho}_v, \underline{\Delta}_{J_v}^{\natural}) \cong X_{\text{tri}, J'-\text{dR}}^{\square}(\bar{\rho}_p, \underline{\Delta}_{J'}^{\natural}) \times_{\widehat{T}(\underline{\Delta}_{J'})} \widehat{T}(\underline{\Delta}_J^{\natural}),$$

and one deduces from Thm.3.22 (applied to $X_p(\bar{\rho}, \underline{\Delta}_{J'})$):

Corollary 3.23. *Keep the situation of Thm.3.22, and let $J' \subseteq J$, the isomorphism (31) (with J replaced by J') induces an isomorphism*

$$\widehat{\mathcal{O}}_{X_p(\bar{\rho}, \underline{\Delta}_J, J'), x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g \times \iota_p^{-1}(X_{\text{tri}, J'-\text{dR}}^{\square}(\bar{\rho}_p, \underline{\Delta}_{J'}^{\natural})), x}.$$

The following corollary would play a crucial role in our proof of the existence of companion points:

Corollary 3.24. *Keep the situation of Thm.3.22, and let $J' \subseteq J$, the following statements are equivalent:*

(i) *the natural projection (induced by the closed embedding $X_p(\bar{\rho}, \underline{\Delta}_J) \hookrightarrow X_p(\bar{\rho}, \underline{\Delta}_J, J')$)*

$$\widehat{\mathcal{O}}_{X_p(\bar{\rho}, \underline{\Delta}_J, J'), x} \twoheadrightarrow \widehat{\mathcal{O}}_{X_p(\bar{\rho}, \underline{\Delta}_J), x}$$

is an isomorphism;

(ii) *$X_p(\bar{\rho}, \underline{\Delta}_J, J')$ is smooth at x ;*

(iii) *$(J \setminus J') \cap \Sigma(x) = \emptyset$.*

Proof. The equivalence of (i) and (ii) is clear since $X_p(\bar{\rho}, \underline{\Delta}_J, J')$ has the same dimension ($g + 4|S| + 3[F^+ : \mathbb{Q}] - 2|J|$) as $X_p(\bar{\rho}, \underline{\Delta}_J)$, and $X_p(\bar{\rho}, \underline{\Delta}_J)$ is smooth at x . As in the proof of Thm.3.22, to prove (ii) is equivalent to (iii), it's sufficient to show that $(J \setminus J') \cap \Sigma(x) = \emptyset$ if and only if

$$\dim_{k(x)} T_{X_p(\bar{\rho}, \underline{\Delta}_J, J'), x} = g + 4|S| + 3[F^+ : \mathbb{Q}] - 2|J|,$$

thus by Cor.3.23, if and only if

$$\dim_{k(x)} T_{\mathfrak{X}_{\overline{\rho}}^{\square} \times \mathbb{U}^g \times \iota_p^{-1}(X_{\text{tri}, J' - \text{dR}}^{\square}(\overline{\rho}_p, \underline{\lambda}_J^{\natural})), x} = g + 4|S| + 3[F^+ : \mathbb{Q}] - 2|J|.$$

However, this follows from Thm.2.5 (2). \square

4. COMPANION POINTS AND LOCAL-GLOBAL COMPATIBILITY

4.1. Breuil's locally analytic socle conjecture. Let $\rho : \text{Gal}_F \rightarrow \text{GL}_2(E)$ be a continuous representation such that $\rho \otimes \varepsilon \cong \rho^{\vee} \circ c$, ρ is unramified outside S , $\overline{\rho}$ is absolutely irreducible. Suppose moreover

- (1) $\widehat{S}(U^p, E)_{\text{alg}, \overline{\rho}}[\mathfrak{m}_{\rho}] \neq 0$;
- (2) $\rho_{\overline{v}} := \rho|_{\text{Gal}_{F_{\overline{v}}}}$ is regular crystalline of distinct Hodge-Tate weights for all $v|p$.

Note by the results in [13], the condition (1) can be replaced by $J_{B_p}(\widehat{S}(U^p, E)^{\text{an}}[\mathfrak{m}_{\rho}]) \neq 0$ (assuming (2), $p > 2$ and $\overline{\rho}|_{\text{Gal}_{F(\zeta_p)}}$ adequate). Let $\underline{\lambda}_{\Sigma_p} := (\lambda_{1,\sigma}, \lambda_{2,\sigma})_{\sigma \in \Sigma_p} \in \mathbb{Z}^{2|\Sigma_p|}$ such that $\text{HT}(\rho_{\overline{v}}) = -\underline{\lambda}_{\Sigma_{\overline{v}}}^{\natural} = -(\lambda_{1,\sigma}, \lambda_{2,\sigma} - 1)_{\sigma \in \Sigma_{\overline{v}}}$ for $v|p$; let $\alpha_{\overline{v},1}, \alpha_{\overline{v},2}$ be the two eigenvalues of the crystalline Frobenius $\varphi^{[F_{\overline{v},0}:\mathbb{Q}_p]}$ on $D_{\text{cris}}(\rho_{\overline{v}})$ for $v|p$. For $s = (s_{\overline{v}})_{v|p} \in \mathcal{S}_2^{|\Sigma_p|}$ (which is in fact the Weyl group of G_p), put $\psi_s := \otimes_{v|p} \text{unr}_{\overline{v}}(q_{\overline{v}}^{-1} \alpha_{\overline{v}, s_{\overline{v}}^{-1}(1)}) \otimes \text{unr}_{\overline{v}}(\alpha_{\overline{v}, s_{\overline{v}}^{-1}(2)})$, and $\delta_s := \psi_s \delta_{\underline{\lambda}_{\Sigma_p}}$, which is a locally algebraic character of T_p ; note since $\underline{\lambda}_{\Sigma_p}$ is dominant, $I(\delta_s \delta_{B_p}^{-1})$ is locally algebraic, moreover, for $s, s' \in \mathcal{S}_2^{|\Sigma_p|}$, $I(\delta_s \delta_{B_p}^{-1}) \cong I(\delta_{s'} \delta_{B_p}^{-1}) =: I_0(\rho_p)$. By classical local Langlands correspondence, there is an injection of locally analytic representations of G_p :

$$I_0(\rho_p) \hookrightarrow \widehat{S}(U^p, E)_{\overline{\rho}}[\mathfrak{m}_{\rho}],$$

which gives (by applying Jacquet-Emerton functor) classical points $z_s = (\mathfrak{m}_{\rho}, \delta_s)$ for all $s \in \mathcal{S}_2^{|\Sigma_p|}$ in $\mathcal{E}(U^p)_{\overline{\rho}}$. For $v|p$, let $\Sigma(z_s)_{\overline{v}} \subset \Sigma_{\overline{v}}$ such that $((\delta_s)_{\Sigma(z_s)}^c)_{\overline{v}}^{\natural}$ (cf. (21), (23)) is a trianguline parameter of $\rho_{\overline{v}}$, and $\Sigma(z_s) := \cup_{v|p} \Sigma(z_s)_{\overline{v}}$.

Conjecture 4.1 (Breuil). *Keep the notation as above, let χ be a continuous character of T_p over E , then $I(\chi) \hookrightarrow \widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathfrak{m}_{\rho}]$ if and only if there exist $s \in \mathcal{S}_2^{|\Sigma_p|}$ and $J \subseteq \Sigma(z_s)_{\overline{v}}$ such that $\chi = (\delta_s)_J^c \delta_{B_p}^{-1}$.*

This conjecture is in fact equivalent to the following conjecture on companion points on the eigenvariety:

Conjecture 4.2. (1) *Let $\chi : T_p \rightarrow E^{\times}$, $(\mathfrak{m}_{\rho}, \chi) \in \mathcal{E}(U^p)_{\overline{\rho}}$ if and only if there exist $s \in \mathcal{S}_2^{|\Sigma_p|}$ and $J \subseteq \Sigma(z_s)_{\overline{v}}$ such that $\chi = (\delta_s)_J^c$.*

(2) *For $s \in \mathcal{S}_2^{|\Sigma_p|}$ and $J \subseteq \Sigma(z_s)$, the point $(z_s)_J^c := (\mathfrak{m}_{\rho}, (\delta_s)_J^c)$ lies moreover in $\mathcal{E}(U^p, \underline{\lambda}_{\Sigma_p \setminus J})_{\overline{\rho}}$.*

First note the “only if” part in (1) of Conj.4.2 is an easy consequence of the global triangulation theory: if $(\mathfrak{m}_{\rho}, \chi) \in \mathcal{E}(U^p)$, by global triangulation theory applied to the point $(\mathfrak{m}_{\rho}, \chi)$, there exists $S \subseteq N(\chi)$ (cf. (22)) such that $((\chi_S^c)_{\overline{v}}^{\natural})$ is a trianguline parameter of $\rho_{\overline{v}}$ for all $v|p$, so there exists $s \in \mathcal{S}_2^{|\Sigma_p|}$ such that $\chi_S^c = (\delta_s)_{\Sigma(z_s)}^c$. Moreover, since $S \subseteq \Sigma_p \setminus N(\chi_S^c)$, $S \subseteq \Sigma(z_s)$, and hence $\chi = (\delta_s)_{\Sigma(z_s) \setminus S}^c$.

We show the equivalence of the above two conjectures. Assuming Conj.4.1, for $s \in \mathcal{S}_2^{|\Sigma_p|}$ and $J \subseteq \Sigma(z_s)$, by applying Jacquet-Emerton functor to an injection

$$I((\delta_s)_J^c \delta_{B_p}^{-1}) \hookrightarrow \widehat{S}(U^p, E)^{\text{an}}[\mathfrak{m}_{\rho}]$$

(which automatically factors through $\widehat{S}(U^p, E)^{\text{an}}(\underline{\lambda}_{\Sigma_p \setminus J})[\mathfrak{m}_{\rho}]$), we would get the point $(z_s)_J^c \in \mathcal{E}(U^p, \underline{\lambda}_{\Sigma_p \setminus J})$. Conversely, assuming Conj.4.2, if $I(\chi) \hookrightarrow \widehat{S}(U^p, E)[\mathfrak{m}_{\rho}]$, applying Jacquet-Emerton functor, we get a point $(\mathfrak{m}_{\rho}, \chi \delta_B) \in \mathcal{E}(U^p)$, thus the “only if” part of Conj.4.1 follows from the “only if” part of Conj.4.2. The

“if” part of Conj.4.1 follows directly from Conj.4.2 (2) and the following bijections (by Prop.B.5 below, and the first bijection is obvious)

$$(32) \quad \begin{aligned} \operatorname{Hom}_{G_p} (I((\delta_s)_J^c \delta_{B_p}^{-1}), \widehat{S}(U^p, E)^{\text{an}}[\mathfrak{m}_\rho]) &\cong \operatorname{Hom}_{G_p} (I((\delta_s)_J^c \delta_{B_p}^{-1}), \widehat{S}(U^p, E)^{\text{an}}(\Delta_{\Sigma_p \setminus J})[\mathfrak{m}_\rho]) \\ &\cong \operatorname{Hom}_{T_p} ((\delta_s)_J^c, J_{B_p}(\widehat{S}(U^p, E)^{\text{an}}(\Delta_{\Sigma_p \setminus J}))[\mathfrak{m}_\rho]) \end{aligned}$$

for all $s \in S_2^{|S_p|}$ and $J \subseteq \Sigma_p$.

In the following, we assume $p > 2$ and $\bar{\rho}|_{\operatorname{Gal}_F(\zeta_p)}$ is adequate. And we prove Conj.4.1 (and hence Conj.4.2) in this case in the next section. We would work with the patched eigenvariety and show a similar result of Conj.4.1 with $\widehat{S}(U^p, E)$ replaced by Π_∞ . Indeed, since $\Pi_\infty[\mathfrak{a}] \cong \widehat{S}(U^p, E)$, if we denote \mathfrak{m}_ρ the maximal ideal of $R_\infty \otimes_{\mathcal{O}_E} E$ corresponding to ρ (via $R_\infty/\mathfrak{a} \xrightarrow{\sim} \mathcal{R}_{S, \bar{\rho}}$), then $\Pi_\infty[\mathfrak{m}_\rho] \cong \widehat{S}(U^p, E)[\mathfrak{m}_\rho]$. For $s \in S_2^{|S_p|}$, one has a point $x_s = (\mathfrak{m}_\rho, \delta_s) \in X_p(\bar{\rho})$, which is moreover classical and lies hence in $X_p(\bar{\rho}, \Delta_{\Sigma_p})$. By $\Pi_\infty[\mathfrak{m}_\rho] \cong \widehat{S}(U^p, E)[\mathfrak{m}_\rho]$ and the adjunction property in (32), one has the following easy lemma

Lemma 4.3. *Keep the above notation, $(x_s)_J^c \in X_p(\bar{\rho}, \Delta_{\Sigma_p \setminus J})$ if and only if $(z_s)_J^c \in \mathcal{E}(U^p, \Delta_{\Sigma_p \setminus J})$.*

4.2. Main results. Let $J \subseteq \Sigma_p$, $\underline{\lambda}_J := (\lambda_{1, \sigma}, \lambda_{2, \sigma})_{\sigma \in J} \in \mathbb{Z}^{2|J|}$ dominant.

Theorem 4.4. *Let $x = (y, \delta)$ be a spherical, very regular point in $X_p(\bar{\rho}, \underline{\lambda}_J)$ with $N(\delta) = J$, and $\rho_{x, \tilde{v}}$ generic for all $v \in S \setminus S_p$. Suppose $\Sigma(x) \neq \emptyset$ (note $\Sigma(x) \subseteq N(\delta) = J$), then for all $\sigma \in \Sigma(x)$, $x_\sigma^c = (y, \delta_\sigma^c) \in X_p(\bar{\rho}, \underline{\lambda}_{J \setminus \{\sigma\}})$ (note $N(\delta_\sigma^c) = J \setminus \{\sigma\}$).*

Remark 4.5. *By Prop.B.5 and the assumption $J = N(\delta)$, one sees $x = (y, \delta) \in X_p(\bar{\rho}, \underline{\lambda}_{N(\delta)})$ is equivalent to the existence of an injection of locally \mathbb{Q}_p -analytic representations of G_p*

$$I(\delta \delta_{B_p}^{-1}) \hookrightarrow \Pi_\infty^{R_\infty - \text{an}}(\underline{\lambda}_{N(\delta)})[\mathfrak{p}_y].$$

Similarly, $x_\sigma^c \in X_p(\bar{\rho}, \underline{\lambda}_{N(\delta) \setminus \{\sigma\}})$ is equivalent to the existence of $I(\delta_\sigma^c \delta_{B_p}^{-1}) \hookrightarrow \Pi_\infty^{R_\infty - \text{an}}(\underline{\lambda}_{N(\delta) \setminus \{\sigma\}})[\mathfrak{p}_y]$.

Proof. Suppose $\Sigma(x) \neq \emptyset$, and let $\sigma \in \Sigma(x)$, $J' := J \setminus \{\sigma\}$, consider the following closed rigid subspaces of $X_p(\bar{\rho})$:

$$X_p(\bar{\rho}, \underline{\lambda}_J) \hookrightarrow X_p(\bar{\rho}, \underline{\lambda}_J, J') \hookrightarrow X_p(\bar{\rho}, \underline{\lambda}_{J'}).$$

By Prop.3.10, there exists an open affinoid neighborhood $U_J := \operatorname{Spm} B_J$ (resp. $U_{J'} := \operatorname{Spm} B_{J'}$) of x in $X_p(\bar{\rho}, \underline{\lambda}_J)$ (resp. in $X_p(\bar{\rho}, \underline{\lambda}_{J'})$), such that

- $\omega_\infty(U_J)$ (resp. $\omega_\infty(U_{J'})$) is an affinoid open, denoted by $\operatorname{Spm} A_J$ (resp. $\operatorname{Spm} A_{J'}$) in $\mathcal{W}_\infty(\underline{\lambda}_J)$ (resp. $\mathcal{W}_\infty(\underline{\lambda}_{J'})$);
- $\kappa_\infty^{-1}(\{\kappa_\infty(x)\}) = \{x\}$, i.e. x is the only point in U_J (resp. $U_{J'}$) lying above $\omega := \omega_\infty(x)$;
- $M_J := \Gamma(U_J, \mathcal{M}_\infty(\underline{\lambda}_J))$ (resp. $M_{J'} := \Gamma(U_{J'}, \mathcal{M}_\infty(\underline{\lambda}_{J'}))$) is a locally free A_J (resp. $A_{J'}$)-module.

One has thus (where \mathfrak{m}_ω denotes the maximal ideal of A_J or $A_{J'}$ corresponding to ω , \mathfrak{m}_x the maximal ideal of B_J or $B_{J'}$ corresponding to x , “[\cdot]” denotes eigenspaces and “[\cdot]{ \cdot ” denotes generalized eigenspaces)

$$\begin{aligned} (M_J/\mathfrak{m}_\omega M_J)^\vee &\xrightarrow{\sim} J_{B_p}(\Pi_\infty^{R_\infty - \text{an}}(\underline{\lambda}_J))[\mathfrak{m}_\omega]\{\mathfrak{m}_x\} \\ \left(\text{resp. } (M_{J'}/\mathfrak{m}_\omega M_{J'})^\vee &\xrightarrow{\sim} J_{B_p}(\Pi_\infty^{R_\infty - \text{an}}(\underline{\lambda}_{J'}))[\mathfrak{m}_\omega]\{\mathfrak{m}_x\} \right). \end{aligned}$$

Claim: The natural injection

$$(33) \quad J_{B_p}(\Pi_\infty^{R_\infty - \text{an}}(\underline{\lambda}_J))[\mathfrak{m}_\omega]\{\mathfrak{m}_x\} \hookrightarrow J_{B_p}(\Pi_\infty^{R_\infty - \text{an}}(\underline{\lambda}_{J'}))[\mathfrak{m}_\omega]\{\mathfrak{m}_x\}$$

is not surjective (if $\sigma \in \Sigma(x) \cap J$).

Assuming the claim, the theorem then follows by applying Breuil's adjunction formula to the right set of (33): Denote by \mathfrak{p}_y (resp. \mathfrak{p}'_y) the prime ideal of R_∞ (resp. of S_∞) corresponding to y , thus

$$J_{B_p}(W)[\mathfrak{m}_\omega]\{\mathfrak{m}_x\} = J_{B_p}(W)[\mathfrak{p}'_y, T_p^0 = \delta]\{\mathfrak{p}_y, T_p = \delta\}$$

for any locally R_∞ -analytic closed subrepresentation W of $\Pi_\infty^{R_\infty-\text{an}}$. Let (by (33))

$$v \in J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'}))[\mathfrak{p}'_y, T_p^0 = \delta]\{\mathfrak{p}_y, T_p = \delta\} \setminus J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_J))[\mathfrak{p}'_y, T_p^0 = \delta]\{\mathfrak{p}_y, T_p = \delta\}$$

and consider the T_p -subrepresentation generated by v , which has the form $\delta_{\text{wt}(\delta)} \otimes_E \pi_{\psi_\delta}$ where π_{ψ_δ} is a finite dimensional unramified smooth representation of T_p with Jordan-Holder factors all isomorphic to $\psi_\delta = \delta \delta_{\text{wt}(\delta)}^{-1}$. By Prop.B.5 one has

$$(34) \quad \text{Hom}_{G_p}(\mathcal{F}_{\overline{B}_p}^{G_p}(M_{J'}(-\text{wt}(\delta))^\vee, \pi_{\psi_\delta} \otimes_E \delta_{B_p}^{-1}), \Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{p}'_y]\{\mathfrak{p}_y\}) \\ \xrightarrow{\sim} \text{Hom}_{T_p}(\pi_{\psi_\delta} \otimes_E \delta_{\text{wt}(\delta)}, J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{p}'_y]\{\mathfrak{p}_y\})).$$

On the other hand, $\mathcal{F}_{\overline{B}_p}^{G_p}(M_{J'}(-\text{wt}(\delta))^\vee, \pi_{\psi_\delta} \otimes_E \delta_{B_p}^{-1})$ sits in an exact sequence (e.g. see Prop.B.4)

$$0 \longrightarrow V_\sigma := \mathcal{F}_{\overline{B}_p}^{G_p}(L(-s_\sigma \cdot \text{wt}(-\delta)), \pi_{\psi_\delta} \otimes_E \delta_{B_p}^{-1}) \longrightarrow \mathcal{F}_{\overline{B}_p}^{G_p}(M_{J'}(-\text{wt}(\delta))^\vee, \pi_{\psi_\delta} \otimes_E \delta_{B_p}^{-1}) \\ \longrightarrow V_0 := \mathcal{F}_{\overline{B}_p}^{G_p}(L(-\text{wt}(\delta)), \pi_{\psi_\delta} \otimes_E \delta_{B_p}^{-1}) \longrightarrow 0,$$

Let $f : \mathcal{F}_{\overline{B}_p}^{G_p}(M_{J'}(-\text{wt}(\delta))^\vee, \pi_{\psi_\delta} \otimes_E \delta_{B_p}^{-1}) \rightarrow \Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{p}'_y]\{\mathfrak{p}_y\}$ be the map in the left set of (34) corresponding to the injection map induced by v in the right set. We see f does not factor through V_0 , since otherwise, $\text{Im}(f)$ would be contained in $\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_J)[\mathfrak{p}'_y]\{\mathfrak{p}_y\}$, and hence $v \in J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_J)[\mathfrak{p}'_y]\{\mathfrak{p}_y\})$ by taking Jacquet-Emerton functor, which contradicts to our choice of v . Thus we see

$$\text{Hom}_{G_p}(V_\sigma, \Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{p}'_y]\{\mathfrak{p}_y\}) = \text{Hom}_{G_p}(V_\sigma, \Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{p}'_y]\{\mathfrak{p}_y\}) \neq 0;$$

since the irreducible components of V_σ are all $I(\delta_\sigma^c \delta_{B_p}^{-1})$, we deduce

$$(35) \quad \text{Hom}_{G_p}(I(\delta_\sigma^c \delta_{B_p}^{-1}), \Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{p}'_y]\{\mathfrak{p}_y\}) \neq 0,$$

by Prop.B.5, the above set can be identified with $J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'}))[\mathfrak{p}'_y, T_p = \delta_\sigma^c]\{\mathfrak{p}_y\}$, which is in particular finite dimensional (since $J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'}))$ is essentially admissible as a $\mathbb{Z}_p^q \times T_p$ -representation, where $S_\infty \cong \mathcal{O}_E[\mathbb{Z}_p^q]$). One can then deduce from (35):

$$\text{Hom}_{G_p}(I(\delta_\sigma^c \delta_{B_p}^{-1}), \Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{p}_y]) \neq 0,$$

the theorem follows.

We prove the claim. Consider the finite free $(A_J)_{\mathfrak{m}_\omega}$ -module (resp. $(A_{J'})_{\mathfrak{m}_\omega}$ -module) $(M_J)_{\mathfrak{m}_\omega}$ (resp. $(M_{J'})_{\mathfrak{m}_\omega}$), let $s_J := \text{rk}_{(A_J)_{\mathfrak{m}_\omega}}(M_J)_{\mathfrak{m}_\omega}$ (resp. $s_{J'} := \text{rk}_{(A_{J'})_{\mathfrak{m}_\omega}}(M_{J'})_{\mathfrak{m}_\omega}$), by the isomorphisms before the claim, it's sufficient to prove $s_J < s_{J'}$.

Since $\mathcal{M}_\infty(\underline{\Delta}_J)$ (resp. $\mathcal{M}_\infty(\underline{\Delta}_{J'})$) is Cohen-Macaulay over $X_p(\overline{\rho}, \underline{\Delta}_J)$ (resp. over $X_p(\overline{\rho}, \underline{\Delta}_{J'})$) by Cor.3.12 (2), and $X_p(\overline{\rho}, \underline{\Delta}_J)$ (resp. $X_p(\overline{\rho}, \underline{\Delta}_{J'})$) is smooth at x by Thm.3.22, by [23, Cor.17.3.5 (i)], shrinking U_J (resp. $U_{J'}$), one can assume M_J (resp. $M_{J'}$) is locally free as a B_J -module (resp. as a $B_{J'}$ -module), and thus $(M_J)_{\mathfrak{m}_\omega} = (M_J)_{\mathfrak{m}_x}$ (resp. $(M_{J'})_{\mathfrak{m}_\omega} = (M_{J'})_{\mathfrak{m}_x}$) is a free $(B_J)_{\mathfrak{m}_\omega} = (B_J)_{\mathfrak{m}_x}$ -module (resp. $(B_{J'})_{\mathfrak{m}_\omega} = (B_{J'})_{\mathfrak{m}_x}$ -module), say of rank r_J (resp. $r_{J'}$); since $(M_J)_{\mathfrak{m}_\omega}$ (resp. $(M_{J'})_{\mathfrak{m}_\omega}$) is free over $(A_J)_{\mathfrak{m}_\omega}$ (resp. $(A_{J'})_{\mathfrak{m}_\omega}$), this implies in particular $(B_J)_{\mathfrak{m}_x}$ (resp. $(B_{J'})_{\mathfrak{m}_x}$), as a direct factor of $(M_J)_{\mathfrak{m}_\omega}$ (resp. of $(M_{J'})_{\mathfrak{m}_\omega}$), is also a free $(A_J)_{\mathfrak{m}_\omega}$ (resp. $(A_{J'})_{\mathfrak{m}_\omega}$)-module, say of rank e_J (resp. $e_{J'}$). It's straightforward to see $s_J = r_J e_J$, $s_{J'} = r_{J'} e_{J'}$.

Since

$$(M_J/\mathfrak{m}_x M_J)^\vee \xrightarrow{\sim} J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_J)[\mathfrak{m}_x]) \\ \left(\text{resp. } (M_{J'}/\mathfrak{m}_x M_{J'})^\vee \xrightarrow{\sim} J_{B_p}(\Pi_\infty^{R_\infty-\text{an}}(\underline{\Delta}_{J'})[\mathfrak{m}_x]) \right),$$

and by the natural injection

$$J_{B_p}(\Pi_\infty^{R-\text{an}}(\underline{\lambda}_J)[\mathfrak{m}_x]) \hookrightarrow J_{B_p}(\Pi_\infty^{R-\text{an}}(\underline{\lambda}_{J'})[\mathfrak{m}_x]),$$

one has $r_J = \dim_{k(x)} M_J/\mathfrak{m}_x M_J \leq \dim_{k(x)} M_{J'}/\mathfrak{m}_x M_{J'} = r_{J'}$. It's thus sufficient to prove $e_J < e_{J'}$, which would follow from Cor.3.24:

Let B'_J (resp. A'_J) be quotient of $B_{J'}$ (resp. of $A_{J'}$) with $\text{Spm } B'_J \cong \text{Spm } B_{J'} \times_{\widehat{T}_p^0(\underline{\lambda}_{J'})} \widehat{T}_p^0(\underline{\lambda}_J)$ (resp. $\text{Spm } A'_J \cong \text{Spm } A_{J'} \times_{\widehat{T}_p^0(\underline{\lambda}_{J'})} \widehat{T}_p^0(\underline{\lambda}_J)$), $M'_J := M_{J'} \otimes_{B_{J'}} B'_J \cong M_{J'} \otimes_{A_{J'}} A'_J$. Since $B_{J'}$ is locally free over $A_{J'}$, B'_J is locally free over A'_J and in particular, $(B'_J)_{\mathfrak{m}_x}$ is a free $(A'_J)_{\mathfrak{m}_\omega}$ -module of rank $e_{J'}$. For a noetherian local E -algebra R , denote by R^\wedge the completion of R at its maximal ideal. One gets in particular complete noetherian local E -algebras $(B_J)_{\mathfrak{m}_x}^\wedge, (B'_J)_{\mathfrak{m}_x}^\wedge, (A_J)_{\mathfrak{m}_\omega}^\wedge, (A'_J)_{\mathfrak{m}_\omega}^\wedge$, which are in fact the complete local algebras of $X_p(\overline{\rho}, \underline{\lambda}_J)$ at x , $X_p(\overline{\rho}, \underline{\lambda}_J, J')$ at x , $\mathcal{W}_\infty(\underline{\lambda}_J)$ at ω , $\mathcal{W}_\infty(\underline{\lambda}_J)$ at ω respectively. In particular, $(A_J)_{\mathfrak{m}_\omega}^\wedge \cong (A'_J)_{\mathfrak{m}_\omega}^\wedge$, and the natural morphisms $X_p(\overline{\rho}, \underline{\lambda}_J) \hookrightarrow X_p(\overline{\rho}, \underline{\lambda}_J, J') \rightarrow \mathcal{W}_\infty(\underline{\lambda}_J)$ induce

$$(A_J)_{\mathfrak{m}_\omega}^\wedge \hookrightarrow (B'_J)_{\mathfrak{m}_x}^\wedge \twoheadrightarrow (B_J)_{\mathfrak{m}_x}^\wedge.$$

By Cor.3.24, the last map is *not* bijective (since $J \setminus J' = \{\sigma\} \subset \Sigma(x)$); since $(B'_J)_{\mathfrak{m}_x}^\wedge$ (resp. $(B_J)_{\mathfrak{m}_x}^\wedge$) is free of rank $e_{J'}$ (resp. e_J) over $(A_J)_{\mathfrak{m}_\omega}^\wedge$, one gets $e_{J'} > e_J$ and hence $s_{J'} > s_J$, the claim (hence the theorem) follows. \square

Corollary 4.6. *Suppose $p > 2$ and $\overline{\rho}|_{\text{Gal}_F(\zeta_p)}$ is adequate, Conj.4.2 (and hence Conj.4.1) is true.*

Proof. With the notation of Conj.4.2, by the discussion following Con.4.2, it's sufficient to show $(z_s)_J^c := (\mathfrak{m}_\rho, (\delta_s)_J^c) \in \mathcal{V}(U^p, \underline{\lambda}_{\Sigma_p \setminus J})$ for all $s \in S_2^{|S_p|}$, $J \subseteq \Sigma(z_s)$. We use the notation as in the end of §4.1, namely, for each $s \in S_2^{|S_p|}$, we have a closed point $x_s = (\mathfrak{m}_\rho, \delta_s)$ in $X_p(\overline{\rho}, \underline{\lambda}_{\Sigma_p})$, note $\Sigma(x_s)$ (defined as in §3.4) is obviously no other than the set $\Sigma(z_s)$ defined in §4.1. Note also $\rho_{\tilde{v}}$ is generic for $v \in S \setminus S_p$. Indeed, ρ corresponds to an automorphic representation π of $G(\mathbb{A}_{F^+})$ with cuspidal strong base change Π to $\text{GL}_2(\mathbb{A}_F)$ (e.g. see [13, Prop.3.4]) which is generic at all finite places of F . Applying Thm.4.4 inductively on $|J|$ (starting with $J = \emptyset$), one sees $(x_s)_J^c \in X_p(\overline{\rho}, \underline{\lambda}_{\Sigma_p \setminus J})$ for all $J \subseteq \Sigma(x_s)$ (note also $\Sigma((x_s)_J^c) = \Sigma(x_s) \setminus J$), which together with Lem.4.3 concludes the proof. \square

We can also deduce from Thm.4.4 some results on the existence of companion points in trianguline case. Let ρ be a continuous representation of Gal_F such that $\rho \otimes \varepsilon \cong \rho^\vee \circ c$, ρ is unramified outside S , $\overline{\rho}$ absolutely irreducible with $\widehat{S}(U^p, E)_{\overline{\rho}}^{\text{alg}} \neq 0$ and $\overline{\rho}|_{\text{Gal}_F(\zeta_p)}$ is adequate. Suppose $J_{B_p}(\widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathfrak{m}_\rho]) \neq 0$, which is equivalent to that ρ is associated to some closed point in $\mathcal{E}(U^p)_{\overline{\rho}}$ and implies in particular $\rho_{\tilde{v}}$ is trianguline for all $v|p$. For $v|p$, let $\chi_{\tilde{v}}$ be a continuous character of $T_{\tilde{v}}$ in E^\times such that $\chi_{\tilde{v}}^{\frac{1}{p}}$ is a trianguline parameter of $\rho_{\tilde{v}}$, we are concerned with the following case

- $\chi_{\tilde{v}}$ is locally algebraic, very regular, and $\chi_{\tilde{v}} \chi_{\text{wt}(\chi_{\tilde{v}})}^{-1}$ is unramified for all $v|p$;
- $\text{wt}(\chi_{\tilde{v}}^{\frac{1}{p}})_{\sigma,1} \neq \text{wt}(\chi_{\tilde{v}}^{\frac{1}{p}})_{\sigma,2}$ (distinct Hodge-Tate weights condition);
- $\rho_{\tilde{v}}$ is *not* crystalline for all $v|p$ (which, together with the above conditions, implies the trianguline parameter $\chi_{\tilde{v}}$ is unique);
- for $v \in S \setminus S_p$, $\rho_{\tilde{v}}$ is generic.

There exist thus a set $\Sigma(\rho)$, and a locally algebraic character δ of T_p with $\text{wt}(\delta)$ dominant such that $\delta_{\Sigma(\rho)_{\tilde{v}}}^c = \chi_{\tilde{v}}$ where $\Sigma(\rho)_{\tilde{v}} := \Sigma(\rho) \cap \Sigma_{\tilde{v}}$ which equals $\{\sigma \in \Sigma_{\tilde{v}} \mid \text{wt}(\chi_{\tilde{v}})_{\sigma,1} < \text{wt}(\chi_{\tilde{v}})_{\sigma,2}\}$. One has

Corollary 4.7. *Keep the notation and assumption as above, let χ be a continuous character of T_p in E^\times , then $(\mathfrak{m}_\rho, \chi) \in \mathcal{E}(U^p)_{\overline{\rho}}$ if and only if there exists $J \subseteq \Sigma(\rho)$ such that $\chi = \delta_J^c$.*

Proof. The “only if” part follows again from global triangulation theory. We prove the existence of companion points. By assumption $J_{B_p}(\widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathfrak{m}_\rho]) \neq 0$ (and the “only if” part), there exists $J \subseteq$

$\Sigma(\rho)$ such that $(\mathbf{m}_\rho, \delta_J^c) \in \mathcal{E}(U^p)_{\overline{\rho}}$, in other words, $J_{B_p}(\widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathbf{m}_\rho])[T_p = \delta_J^c] \neq 0$. Applying Breuil's adjunction formula, we see (where $\psi_\delta = \delta\delta_{\text{wt}(\delta)}^{-1}$)

$$\text{Hom}_{G_p} \left(\mathcal{F}_{B_p}^{G_p}(M(-\text{wt}(\delta_J^c))^\vee, \psi_\delta \delta_{B_p}^{-1}), \widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathbf{m}_\rho] \right) = J_{B_p}(\widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathbf{m}_\rho])[T_p = \delta_J^c] \neq 0;$$

Thus there exists $S \subseteq \Sigma_p \setminus J$ such that (since the irreducible components of $\mathcal{F}_{B_p}^{G_p}(M(-\text{wt}(\delta_J^c))^\vee, \psi_\delta \delta_{B_p}^{-1})$ are given by $\{I(\delta_{J \cup S}^c \delta_{B_p}^{-1})\}_{S \subseteq \Sigma_p \setminus J}$, where the irreducibility follows from the assumption on $\chi_{\tilde{v}}$)

$$I(\delta_{J \cup S}^c \delta_{B_p}^{-1}) \hookrightarrow \widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathbf{m}_\rho] \cong \Pi_\infty^{R_\infty - \text{an}}[\mathbf{m}_\rho]$$

where we also use \mathbf{m}_ρ to denote the maximal ideal of $R_\infty[1/p]$ associated to ρ . We thus get a point $z_{J \cup S}^c$ (resp. $x_{J \cup S}^c$) in $\mathcal{E}(U^p, \text{wt}(\delta)_{\Sigma_p \setminus (J \cup S)})_{\overline{\rho}}$ (resp. in $X_p(\overline{\rho}, \text{wt}(\delta)_{\Sigma_p \setminus (J \cup S)})$). By global triangulation theory applied to $z_{J \cup S}^c$ (or by Thm.3.21 and consider the image of $x_{J \cup S}^c$ in $X_{\text{tri}}^\square(\overline{\rho}_p)$), we have $J \cup S \subseteq \Sigma(\rho)$; note also the point $x_{J \cup S}^c$ satisfies the hypothesis in Thm.4.4. Thus by Thm.4.4 (inductively), we get $x_{J'}^c \in X_p(\overline{\rho}, \text{wt}(\delta)_{\Sigma_p \setminus J'})$ for all $S \cup J \subseteq J' \subseteq \Sigma(\rho)$. Using a similar result as in Lem.4.3, one gets $z_{J'}^c := (\mathbf{m}_\rho, \delta_{J'}^c) \in \mathcal{E}(U^p, \text{wt}(\delta)_{\Sigma_p \setminus J'})_{\overline{\rho}}$ for all $S \cup J \subseteq J' \subseteq \Sigma(\rho)$. The direction that $z_{\Sigma(\rho)}^c \in \mathcal{E}(U^p)_{\overline{\rho}} \Rightarrow z_J^c \in \mathcal{E}(U^p)_{\overline{\rho}}$ for all $J \subseteq \Sigma(\rho)$ follows from [11, Prop.8.1 (ii)] (see [13, Thm.5.1] for patched eigenvariety case), which allows to conclude. \square

At last, we propose a locally analytic socle conjecture in this case. Denote by $C(\rho)$ the maximal subset of Σ_p such that $\rho_{\tilde{v}}$ is $C(\rho) \cap \Sigma_{\tilde{v}}$ -de Rham for all $v|p$. Note by [19, PropA.3], $C(\rho) \supseteq \Sigma_p \setminus \Sigma(\rho)$.

Conjecture 4.8. *Keep the notation and assumption as above, $I(\chi \delta_{B_p}^{-1}) \hookrightarrow \widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathbf{m}_\rho]$ if and only if $\chi = \delta_J^c$ for $\Sigma_p \setminus C(\rho) \subseteq J \subseteq \Sigma(\rho)$.*

The “only if” part is known: since $I(\chi \delta_{B_p}^{-1}) \hookrightarrow \widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathbf{m}_\rho]$ implies $(\mathbf{m}_\rho, \chi) \in \mathcal{E}(U^p)_{\overline{\rho}}$, by Cor.4.7, $\chi = \delta_J^c$ for $J \subseteq \Sigma(\rho)$; on the other hand if $I(\delta_J^c \delta_{B_p}^{-1}) \hookrightarrow \widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}[\mathbf{m}_\rho]$, which automatically factors through $\widehat{S}(U^p, E)_{\overline{\rho}}^{\text{an}}(\text{wt}(\delta)_{\Sigma_p \setminus J})[\mathbf{m}_\rho]$, then $(\mathbf{m}_\rho, \delta_J^c)$ lies moreover in $\mathcal{E}(U^p, \text{wt}(\delta)_{\Sigma_p \setminus J})_{\overline{\rho}}$, by Thm.3.7, $\rho_{\tilde{v}}$ is $\Sigma_{\tilde{v}} \setminus (J \cap \Sigma_{\tilde{v}})$ -de Rham for all $v|p$, hence $J \supseteq \Sigma_p \setminus C(\rho)$. By Cor.4.7, the unproven part of Conj.3.7 is thus the following (which can be viewed as a partial classicality):

Conjecture 4.9. *Keep the notation and assumption as above, for $\Sigma_p \setminus C(\rho) \subseteq J \subseteq \Sigma(\rho)$, the point $(\mathbf{m}_\rho, \delta_J^c)$ lies moreover in $\mathcal{E}(U^p, \text{wt}(\delta)_{\Sigma_p \setminus J})_{\overline{\rho}}$.*

APPENDIX A. PARTIALLY DE RHAM B -PAIRS

Recall some results on B -pairs. Let L be a finite extension of \mathbb{Q}_p , $\mathcal{G}_L := \text{Gal}(\overline{L}/L)$, E a finite extension of \mathbb{Q}_p containing all the embeddings of L in $\overline{\mathbb{Q}_p}$. Let A be an artinian local E -algebra, recall (cf. [6, §2], [27, Def.2.11] and [19, Def.1.3])

Definition A.1. (1) *An A - B -pair W is a pair (W_e, W_{dR}^+) where W_e is a free $B_e \otimes_{\mathbb{Q}_p} A$ -module, and W_{dR}^+ is a Gal_L -invariant $B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} A$ -lattice in $W_{\text{dR}} := W_e \otimes_{B_e} B_{\text{dR}}$. The rank of W is defined to be the rank of W_e over $B_e \otimes_{\mathbb{Q}_p} A$.*

(2) *An A - B -pair W is called J -de Rham if $W_{\text{dR}, \sigma}^{\text{Gal}_L}$ is a free A -module of rank $\text{rk } W$ for all $\sigma \in J$ (where $W_{\text{dR}} \cong \bigoplus_{\sigma \in \Sigma_L} W_{\text{dR}, \sigma}$ with respect to the isomorphism $B_{\text{dR}} \otimes_{\mathbb{Q}_p} A \cong \bigoplus_{\sigma \in \Sigma_L} B_{\text{dR}, \sigma}$, $B_{\text{dR}, \sigma} := B_{\text{dR}} \otimes_{L, \sigma} A$).*

Example A.2. (1) *Let V be a continuous representation of Gal_L over A , one can associate to V an A - B -pair: $W(V) = (B_e \otimes_{\mathbb{Q}_p} V, B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V)$; for $J \subseteq \Sigma_L$, by definition, $W(V)$ is J -de Rham if and only if V is J -de Rham.*

(2) Let $\delta : L^\times \rightarrow A^\times$ be a continuous character, one can associate as in [27, §2.1.2] to δ a rank 1 A - B -pair denoted by $B_A(\delta)$. In fact, by [27, Prop.2.16], for any rank one A - B -pair W , there exists $\delta : L^\times \rightarrow A^\times$ such that $W \cong B_A(\delta)$.

(3) Let $\delta : L^\times \rightarrow E^\times$ be a continuous character, $J \subset \Sigma_L$, by [19, Lem.A.1], $B_E(\delta)$ is J -de Rham if and only if $\text{wt}(\delta)_\sigma \in \mathbb{Z}$ for all $\sigma \in J$.

(4) Let $\delta : L^\times \rightarrow (E[\epsilon]/\epsilon^2)^\times$ be a continuous character, $J \subset \Sigma_L$, by [19, Lem.1.15], $B_{E[\epsilon]/\epsilon^2}(\delta)$ is J -de Rham if and only if $\text{wt}(\delta)_\sigma \in \mathbb{Z}$ for all $\sigma \in J$; let $\bar{\delta} := \delta \pmod{\epsilon} : L^\times \rightarrow E^\times$, thus for any $\sigma \in \Sigma_L$, there exists $d_\sigma \in E$ such that $\text{wt}(\delta)_\sigma = \text{wt}(\bar{\delta})_\sigma + d_\sigma \epsilon$, so $B_{E[\epsilon]/\epsilon^2}(\delta)$ is J -de Rham if and only if $B_E(\bar{\delta})$ is J -de Rham and $d_\sigma = 0$ for all $\sigma \in J$.

For an A - B -pair W , denote by $C^\bullet(W)$ the Gal_L -complex: $[W_e \oplus W_{\text{dR}}^+ \xrightarrow{(x,y) \mapsto x-y} W_{\text{dR}}]$. Following [27, App.], let $H^i(\text{Gal}_L, W) := H^i(\text{Gal}_L, C^\bullet(W))$. By definition, one has an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\text{Gal}_L, W) \rightarrow W_e^{\text{Gal}_L} \oplus (W_{\text{dR}}^+)^{\text{Gal}_L} \rightarrow W_{\text{dR}}^{\text{Gal}_L} \\ \rightarrow H^1(\text{Gal}_L, W) \rightarrow H^1(\text{Gal}_L, W_e) \oplus H^1(\text{Gal}_L, W_{\text{dR}}^+) \rightarrow H^1(\text{Gal}_L, W_{\text{dR}}). \end{aligned}$$

One can (and does) identify $H^1(\text{Gal}_L, W)$ with the group of extensions of A - B -pairs $\text{Ext}^1(B_A, W)$. If $W \cong W(V)$ for some continuous representation V of Gal_L over A , then we have canonical isomorphisms $H^i(\text{Gal}_L, W) \cong H^i(\text{Gal}_L, W(V))$. For $J \subseteq \Sigma_L$, consider the following morphism $C^\bullet(W) \rightarrow [W_e \rightarrow 0] \rightarrow [W_{\text{dR}} \rightarrow 0] \rightarrow [W_{\text{dR}, J} \rightarrow 0]$ where $W_{\text{dR}, J} := \bigoplus_{\sigma \in J} W_{\text{dR}, \sigma}$, and the last map is the natural projection. As in [19], put $H_{g, J}^1(\text{Gal}_L, W) := \text{Ker}(H^1(\text{Gal}_L, W) \rightarrow H^1(\text{Gal}_L, W_{\text{dR}, J}))$. It's obviously that $H_{g, J_1}^1(\text{Gal}_L, W) \cap H_{g, J_2}^1(\text{Gal}_L, W) = H_{g, J_1 \cup J_2}^1(\text{Gal}_L, W)$. Moreover, if W is J -de Rham, for $[W'] \in H^1(\text{Gal}_L, W)$, $[W'] \in H_{g, J}^1(\text{Gal}_L, W)$ if and only if W' is J -de Rham. Note that a morphism of E - B -pairs: $f : W \rightarrow W'$ induces a natural morphism $H_{g, J}^1(\text{Gal}_L, W) \rightarrow H_{g, J}^1(\text{Gal}_L, W')$.

For an E - B -pair W , an algebraic character $\delta = \prod_{\sigma \in \Sigma_L} \sigma^{\text{wt}(\delta)_\sigma}$ of L^\times over E , denote by $W(\delta) := W \otimes B_E(\delta)$ (recall the tensor product $W_1 \otimes W_2$ of E - B -pairs is defined to be the pair $(W_{1, e} \otimes_{B_e} W_{2, e}, W_{1, \text{dR}}^+ \otimes_{B_{\text{dR}}}^+ W_{2, \text{dR}}^+)$). One has in fact

$$W(\delta)_e \cong W_e, \quad W(\delta)_{\text{dR}, \sigma}^+ \cong t^{\text{wt}(\delta)_\sigma} W_{\text{dR}, \sigma}^+, \quad \forall \sigma \in \Sigma_L.$$

For algebraic characters δ_1, δ_2 with $\text{wt}(\delta_1) \geq \text{wt}(\delta_2)$, one has a natural morphism $i = (i_e, i_{\text{dR}}^+) : W(\delta_1) \rightarrow W(\delta_2)$ with $i_e : W(\delta_1)_e \xrightarrow{\sim} W(\delta_2)_e$, and $i_{\text{dR}}^+ : W(\delta_1)_{\text{dR}}^+ \hookrightarrow W(\delta_2)_{\text{dR}}^+$ the natural injection. One gets thus an exact sequence of Gal_L -complexes:

$$\begin{aligned} (36) \quad 0 \longrightarrow [W(\delta_1)_e \oplus W(\delta_1)_{\text{dR}}^+ \rightarrow W(\delta_1)_{\text{dR}}] \longrightarrow [W(\delta_2)_e \oplus W(\delta_2)_{\text{dR}}^+ \rightarrow W(\delta_2)_{\text{dR}}] \\ \longrightarrow [\bigoplus_{\sigma \in \Sigma_L} W(\delta_2)_{\text{dR}, \sigma}^+ / t^{\text{wt}(\delta_1)_\sigma - \text{wt}(\delta_2)_\sigma} \rightarrow 0] \rightarrow 0. \end{aligned}$$

Thus $H^0(\text{Gal}_L, W(\delta_1)) \xrightarrow{\sim} H^0(\text{Gal}_L, W(\delta_2))$ if $H^0(\text{Gal}_L, W(\delta_2)_{\text{dR}, \sigma}^+ / t^{\text{wt}(\delta_1)_\sigma - \text{wt}(\delta_2)_\sigma}) = 0$ for all $\sigma \in J$. Consider now $H^2(\text{Gal}_L, W(\delta)) \cong H^0(\text{Gal}_L, W^\vee(\delta^{-1} \chi_{\text{cyc}}))^\vee$, where the first “ \vee ” denotes the dual of E - B -pairs, and the second is the dual of E -vector spaces and the isomorphism follows from Tate duality. Suppose W is J -de Rham, let $\delta = \prod_{\sigma \in J} \sigma^{k_\sigma}$ be an algebraic character of L^\times with $k_\sigma \in \mathbb{Z}_{\geq 0}$ such that $H^0(\text{Gal}_L, W(\delta)_{\text{dR}, \sigma}^+) = 0$ for all $\sigma \in J$, in other words, the generalized Hodge-Tate weights of $W(\delta)$ are negative for $\sigma \in J$ (which holds when $\text{wt}(\delta)_\sigma$ for all $\sigma \in J$ are sufficiently large). Put

$$(37) \quad \tilde{H}^2(\text{Gal}_L, W) := H^2(\text{Gal}_L, W(\delta)),$$

which is in fact independent of the choice of δ . Indeed, for δ_1, δ_2 algebraic characters of L^\times satisfying the above assumptions (for δ), suppose $\text{wt}(\delta_1)_\sigma \geq \text{wt}(\delta_2)_\sigma$ for all $\sigma \in J$ (the general case can be easily reduced to this case), the Tate dual of the natural map $H^2(\text{Gal}_L, W(\delta_1)) \rightarrow H^2(\text{Gal}_L, W(\delta_2))$ is thus

$H^0(\text{Gal}_L, W^\vee(\delta_2^{-1}\chi_{\text{cyc}})) \hookrightarrow H^0(\text{Gal}_L, W^\vee(\delta_1^{-1}\chi_{\text{cyc}}))$, which is in fact bijective by (36), since by the assumption $H^0(\text{Gal}_L, W(\delta_i)_{\text{dR},\sigma}^+) = 0$ for all $\sigma \in J$, $i = 1, 2$, we have

$$H^0(\text{Gal}_L, (W^\vee(\delta_1^{-1}\chi_{\text{cyc}}))_{\text{dR},\sigma}^+/t^{k_1,\sigma-k_2,\sigma}) = 0$$

for all $\sigma \in J$. Note also that one has a natural projection $\tilde{H}_J^2(\text{Gal}_L, W) \rightarrow H^2(\text{Gal}_L, W)$. By the exact sequence [19, (7)], one has in fact

Proposition A.3. *Let W be an E - B -pair, then*

$$(38) \quad \dim_E H_{g,J}^1(G_L, W) = [L : \mathbb{Q}_p] \text{rk } W + \dim_E H^0(\text{Gal}_L, W) + \dim_E \tilde{H}_J^2(\text{Gal}_L, W) - \dim_E H^0(\text{Gal}_L, W_{\text{dR},J}^+).$$

Proof. We use the notation of *loc. cit.*, and we have

$$\dim_E H_{g,J}^1(G_L, W) = \dim_E H^1(\text{Gal}_L, W(\delta)) + \dim_E H^0(\text{Gal}_L, W) - \dim_E H^0(\text{Gal}_L, W_{\text{dR},J}^+);$$

however, by assumption on δ , we have $H^0(\text{Gal}_L, W(\delta)) = 0$, $H^2(\text{Gal}_L, W(\delta)) \cong \tilde{H}_J^2(\text{Gal}_L, W)$, thus $\dim_E H^1(\text{Gal}_L, W(\delta)) = [L : \mathbb{Q}_p] \text{rk } W + \dim_E \tilde{H}_J^2(\text{Gal}_L, W)$, the proposition follows. \square

Corollary A.4. *Let W be an E - B -pair, if $\tilde{H}^2(\text{Gal}_L, W) = H^2(\text{Gal}_L, W)$, then $\dim_E H_{g,J}^1(\text{Gal}_L, W) = \dim_E H^1(\text{Gal}_L, W) - \dim_E H^0(\text{Gal}_L, W_{\text{dR},J}^+)$.*

Proposition A.5. *Given an exact sequence of E - B -pairs*

$$(39) \quad 0 \rightarrow W_1 \rightarrow W_2 \rightarrow W_3 \rightarrow 0,$$

suppose W_i are J -de Rham for all $i = 1, 2, 3$ and $\tilde{H}_J^2(\text{Gal}_L, W_1) = 0$ (which implies in particular $H^2(\text{Gal}_L, W_1) = 0$), then (39) induces a long exact sequence

$$(40) \quad 0 \rightarrow H^0(\text{Gal}_L, W_1) \rightarrow H^0(\text{Gal}_L, W_2) \rightarrow H^0(\text{Gal}_L, W_3) \rightarrow H_{g,J}^1(\text{Gal}_L, W_1) \rightarrow H_{g,J}^1(\text{Gal}_L, W_2) \rightarrow H_{g,J}^1(\text{Gal}_L, W_3) \rightarrow 0.$$

Proof. Since $H^2(\text{Gal}_L, W_1) = 0$, (39) induces a long exact sequence

$$0 \rightarrow H^0(\text{Gal}_L, W_1) \rightarrow H^0(\text{Gal}_L, W_2) \rightarrow H^0(\text{Gal}_L, W_3) \rightarrow H^1(\text{Gal}_L, W_1) \rightarrow H^1(\text{Gal}_L, W_2) \rightarrow H^1(\text{Gal}_L, W_3) \rightarrow 0.$$

Since W_i are J -de Rham for all $i = 1, 2, 3$, the exact sequence

$$0 \rightarrow W_{1,\text{dR},J} \rightarrow W_{2,\text{dR},J} \rightarrow W_{3,\text{dR},J} \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow H^1(\text{Gal}_L, W_{1,\text{dR},J}) \rightarrow H^1(\text{Gal}_L, W_{2,\text{dR},J}) \rightarrow H^1(\text{Gal}_L, W_{3,\text{dR},J}) \rightarrow 0;$$

moreover, the following diagram commutes:

$$\begin{array}{ccccccccc} H^0(\text{Gal}_L, W_3) & \longrightarrow & H^1(\text{Gal}_L, W_1) & \longrightarrow & H^1(\text{Gal}_L, W_2) & \longrightarrow & H^1(\text{Gal}_L, W_3) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(\text{Gal}_L, W_{3,\text{dR},J}) & \longrightarrow & H^1(\text{Gal}_L, W_{1,\text{dR},J}) & \longrightarrow & H^1(\text{Gal}_L, W_{2,\text{dR},J}) & \longrightarrow & H^1(\text{Gal}_L, W_{3,\text{dR},J}) & \longrightarrow & 0 \end{array}$$

From which we get the exact sequence (40) except the last map (note that the map $H^0(\text{Gal}_L, W_{3,\text{dR},J}) \rightarrow H^1(\text{Gal}_L, W_{1,\text{dR},J})$ equals zero, since W_2 is J -de Rham). Thus it's sufficient to show the induced map $H_{g,J}^1(\text{Gal}_L, W_2) \rightarrow H_{g,J}^1(\text{Gal}_L, W_3)$ is surjective and we prove it by dimension calculation. By Prop. A.3,

$$\begin{aligned} \dim_E H_{g,J}^1(\text{Gal}_L, W_i) &= d_L \text{rk } W_i + \dim_E \tilde{H}_J^2(\text{Gal}_L, W_i) + \dim_E H^0(\text{Gal}_L, W_i) \\ &\quad - \dim_E H^0(\text{Gal}_L, W_{i,\text{dR},J}^+). \end{aligned}$$

Since $\tilde{H}_J^2(\text{Gal}_L, W_1) = 0$, we have $\tilde{H}_J^2(\text{Gal}_L, W_2) \cong \tilde{H}_J^2(\text{Gal}_L, W_3)$. Indeed, let δ be an algebraic character of L^\times over E with $\text{wt}(\delta)_\sigma \in \mathbb{Z}_{\geq 0}$ for $\sigma \in J$ and $\text{wt}(\delta)_\sigma = 0$ for $\sigma \notin J$ satisfying that $H^0(\text{Gal}_L, W_2(\delta)_{\text{dR}, J}^+) = 0$ (thus $H^0(\text{Gal}_L, W_i(\delta)_{\text{dR}, J}^+) = 0$ for all $i = 1, 2, 3$); one gets an isomorphism $H^2(\text{Gal}_L, W_2(\delta)) \cong H^2(\text{Gal}_L, W_3(\delta))$ (hence the precedent isomorphism) from the exact sequence of E - B -pairs $0 \rightarrow W_1(\delta) \rightarrow W_2(\delta) \rightarrow W_3(\delta) \rightarrow 0$, since $H^2(\text{Gal}_L, W_1(\delta)) = 0$ by assumption. Since W_i are all J -de Rham, $\dim_E H^0(\text{Gal}_L, W_{1, \text{dR}, J}^+) + \dim_E H^0(\text{Gal}_L, W_{3, \text{dR}, J}^+) = \dim_E H^0(\text{Gal}_L, W_{2, \text{dR}, J}^+)$. Combining the above calculation, we see

$$\begin{aligned} \dim_E H^0(\text{Gal}_L, W_1) + \dim_{k(x)} H^0(\text{Gal}_L, W_3) + \dim_{k(x)} H_{g, J}^1(\text{Gal}_L, W_2) \\ = \dim_E H^0(\text{Gal}_L, W_2) + \dim_{k(x)} H_{g, J}^1(\text{Gal}_L, W_1) + \dim_{k(x)} H_{g, J}^1(\text{Gal}_L, W_3). \end{aligned}$$

which allows to conclude. \square

APPENDIX B. SOME LOCALLY ANALYTIC REPRESENTATION THEORY

Let V be a locally \mathbb{Q}_p -analytic representation of G_p over E , $J \subseteq \Sigma_p$, a vector $v \in V$ is called *locally J -analytic*, if the induced action of \mathfrak{g}_{Σ_p} on v factors through \mathfrak{g}_J ; we denote by $V^{J-\text{an}}$ the subrepresentation of V of locally J -analytic vectors and V is called *locally J -analytic* if $V = V^{J-\text{an}}$. A vector $v \in V$ is called *$U(\mathfrak{g}_J)$ -finite* (or *J -classical*) if the E -vector space $U(\mathfrak{g}_J)v$ is finite dimensional, and V is called *$U(\mathfrak{g}_J)$ -finite* if all the vectors in V are $U(\mathfrak{g}_J)$ -finite. In particular, V is $U(\mathfrak{g}_{\Sigma_p \setminus J})$ -finite if V is locally J -analytic. Note also that $U(\mathfrak{g}_{\Sigma_p})$ -finite is equivalent to locally algebraic.

As in [18, Prop.5.1.3], one has

Proposition B.1. *Let V be a locally \mathbb{Q}_p -analytic representation of G_p over E , $J \subseteq \Sigma_p$, W an irreducible algebraic representation of G_p over E which is moreover locally J -analytic, then the following composition*

$$(V \otimes_E W)^{\Sigma_p \setminus J-\text{an}} \otimes_E W' \longrightarrow V \otimes_E W \otimes_E W' \longrightarrow V, \quad v \otimes w \otimes w' \mapsto w'(w)v,$$

is injective.

Let $J \subseteq \Sigma_p$, $\underline{\lambda}_J = (\lambda_{1, \sigma}, \lambda_{2, \sigma})_{\sigma \in J} \in \mathbb{Z}^{2|J|}$ be a dominant integral weight of \mathfrak{t}_p , and $L(\underline{\lambda}_J)$ the irreducible algebraic representation of G_p with highest weight $\underline{\lambda}_J$. Let V be a locally \mathbb{Q}_p -analytic representation of G_p over E , put $V(\underline{\lambda}_J) := (V \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J-\text{an}} \otimes_E L(\underline{\lambda}_J)$.

Corollary B.2. *$V(\underline{\lambda}_J)$ is a subrepresentation of V ; if V is moreover admissible, then $V(\underline{\lambda}_J)$ is a closed admissible subrepresentation of V .*

Proof. The first statement follows directly from Prop. B.1. If V is admissible, so is $V \otimes_E L(\underline{\lambda}_J)' \otimes_E L(\underline{\lambda}_J)$. Since $V(\underline{\lambda}_J)$ is obviously a closed subrepresentation of $V \otimes_E L(\underline{\lambda}_J)' \otimes_E L(\underline{\lambda}_J)$, by [29, Prop.6.4], $V(\underline{\lambda}_J)$ is also admissible. By *loc. cit.*, the map $V(\underline{\lambda}_J) \hookrightarrow V$ is strict and has closed image, which concludes the proof. \square

We have moreover the following easy lemma.

Lemma B.3. (1) *An morphism $V \rightarrow W$ of locally \mathbb{Q}_p -analytic representations induces $V(\underline{\lambda}_J) \rightarrow W(\underline{\lambda}_J)$.*

(2) *Suppose there exists a locally $\Sigma_p \setminus J$ -analytic representation W such that $V \cong W \otimes_E L(\underline{\lambda}_J)$, then $V(\underline{\lambda}_J) \cong V$.*

(3) *Let $J' \subseteq J$, then $V(\underline{\lambda}_J)$ is a subrepresentation of $V(\underline{\lambda}_{J'})$.*

Proof. (1) is obvious.

For (2), it's sufficient to prove $(W \otimes_E L(\underline{\lambda}_J) \otimes_E L(\underline{\lambda}_{J'})^{\Sigma_p \setminus J-\text{an}} \cong W$. Note taking locally $\Sigma_p \setminus J$ -analytic vectors is the same as taking $U(\mathfrak{g}_J)$ -invariant vectors. One has however $(W \otimes_E L(\underline{\lambda}_J)' \otimes_E L(\underline{\lambda}_J))^{\text{U}(\mathfrak{g}_J)} \cong W \otimes_E \text{End}_E(L(\underline{\lambda}_J))^{\text{U}(\mathfrak{g}_J)} \cong W \otimes_E \text{End}_{\mathfrak{g}_J}(L(\underline{\lambda}_J)) \cong W$, where the first isomorphism is from the fact that W is locally $\Sigma_p \setminus J$ -analytic, and the last one from the irreducibility of $L(\underline{\lambda}_J)$ as a representation of \mathfrak{g}_J .

For (3), one has

$$\begin{aligned} V(\underline{\lambda}_J) &\cong (V \otimes_E L(\underline{\lambda}_J)')^{\Sigma_p \setminus J-\text{an}} \otimes_E L(\underline{\lambda}_J) \\ &\cong (V \otimes_E L(\underline{\lambda}_{J'})' \otimes_E L(\underline{\lambda}_{J \setminus J'})')^{\Sigma_p \setminus J-\text{an}} \otimes_E L(\underline{\lambda}_{J'}) \otimes_E L(\underline{\lambda}_{J \setminus J'}) \\ &\cong ((V \otimes_E L(\underline{\lambda}_{J'})')^{\Sigma_p \setminus J'-\text{an}} \otimes_E L(\underline{\lambda}_{J \setminus J'})')^{\Sigma_p \setminus (J \setminus J')-\text{an}} \otimes_E L(\underline{\lambda}_{J'}) \otimes_E L(\underline{\lambda}_{J \setminus J'}) \\ &\cong ((V \otimes_E L(\underline{\lambda}_{J'}')^{\Sigma_p \setminus J'-\text{an}} \otimes_E L(\underline{\lambda}_{J'}) \otimes_E L(\underline{\lambda}_{J \setminus J'})')^{\Sigma_p \setminus (J \setminus J')-\text{an}} \otimes_E L(\underline{\lambda}_{J \setminus J'}) \\ &\cong V(\underline{\lambda}_J')(\underline{\lambda}_{J \setminus J'}), \end{aligned}$$

where the third isomorphism follows from the fact that $L(\underline{\lambda}_{J \setminus J'})$ is locally $\Sigma_p \setminus J'$ -analytic, and the fourth from that $L(\underline{\lambda}_{J'})$ is locally $\Sigma_p \setminus (J \setminus J')$ -analytic. From which, together with Cor. B.2, (3) follows. \square

Following [28], to any object M in the BGG category $\mathcal{O}^{\bar{\mathfrak{b}}_{\Sigma_p}}$, and any finite length smooth representation π of T_p over E , can be associated a locally \mathbb{Q}_p -analytic representation $\mathcal{F}_{\bar{B}_p}^{G_p}(M, \pi)$ of G_p over E . The functor $\mathcal{F}_{\bar{B}_p}^{G_p}(\cdot, \cdot)$ is exact in both arguments, covariant for finite length smooth representations of T_p , and contravariant for $\mathcal{O}^{\bar{\mathfrak{b}}_{\Sigma_p}}$ (cf. [28, Thm]).

Let $\underline{\lambda}_{\Sigma_p} = (\lambda_{1,\sigma}, \lambda_{2,\sigma})_{\sigma \in \Sigma_p} \in E^{2|\Sigma_p|}$ be a weight of \mathfrak{t}_p , denote by $M(\underline{\lambda}_{\Sigma_p}) := U(\mathfrak{g}_{p,\Sigma_p}) \otimes_{U(\bar{\mathfrak{b}}_{p,\Sigma_p})} \underline{\lambda}_{\Sigma_p} \cong \otimes_{\sigma \in \Sigma_p} U(\mathfrak{g}_{p,\sigma}) \otimes_{U(\bar{\mathfrak{b}}_{p,\sigma})} \underline{\lambda}_{\sigma} = \otimes_{\sigma \in \Sigma_p} M(\underline{\lambda}_{\sigma})$ the Verma module of highest weight $\underline{\lambda}_{\Sigma_p}$, which admits a unique simple quotient denoted by $L(\underline{\lambda}_{\Sigma_p})$. Suppose moreover $\underline{\lambda}_{\Sigma_p}$ is integral (i.e. $\lambda_{i,\sigma} \in \mathbb{Z}$ for all $i = 1, 2, \sigma \in \Sigma_p$), and put

$$N(\underline{\lambda}) := \{\sigma \in \Sigma_p \mid \lambda_{1,\sigma} \geq \lambda_{2,\sigma}\}.$$

For $\sigma \in \Sigma_p$, $M(\underline{\lambda}_{\sigma})$ is irreducible if $\sigma \notin N(\underline{\lambda})$; for $\sigma \in N(\underline{\lambda})$, $M(\underline{\lambda}_{\sigma})$ lies in an exact sequence

$$0 \rightarrow L(s_{\sigma} \cdot \underline{\lambda}_{\sigma}) \rightarrow M(\underline{\lambda}_{\sigma}) \rightarrow L(\underline{\lambda}_{\sigma}) \rightarrow 0.$$

From which we deduce $M(\underline{\lambda}_{\Sigma_p})$ admits a decreasing filtration (of objects in $\mathcal{O}^{\bar{\mathfrak{b}}_{\Sigma_p}}$):

$$0 = \text{Fil}^{|N(\underline{\lambda})|+1} M(\underline{\lambda}_{\Sigma_p}) \subset \text{Fil}^{|N(\underline{\lambda})|} M(\underline{\lambda}_{\Sigma_p}) \subset \dots \subset \text{Fil}^0 M(\underline{\lambda}_{\Sigma_p}) = M(\underline{\lambda}_{\Sigma_p})$$

such that

$$\text{Fil}^i M(\underline{\lambda}_{\Sigma_p}) / \text{Fil}^{i+1} M(\underline{\lambda}_{\Sigma_p}) \cong \oplus_{J \subseteq N(\underline{\lambda}), |J|=i} L(s_J \cdot \underline{\lambda}_{\Sigma_p}).$$

In fact, one has

$$\text{Fil}^i M(\underline{\lambda}_{\Sigma_p}) = \text{Im} \left(\oplus_{J \subseteq N(\underline{\lambda}), |J|=i} M(s_J \cdot \underline{\lambda}_{\Sigma_p}) \rightarrow M(\underline{\lambda}_{\Sigma_p}) \right).$$

For $J \subseteq N(\underline{\lambda})$, denote by $M_J(\underline{\lambda}_{\Sigma_p}) := M(\underline{\lambda}_{\Sigma_p}) / \text{Im} \left(\oplus_{\sigma \in J} M(s_{\sigma} \cdot \underline{\lambda}_{\Sigma_p}) \rightarrow M(\underline{\lambda}_{\Sigma_p}) \right)$, which is in fact the maximal $U(\mathfrak{g}_J)$ -finite quotient of $M(\underline{\lambda}_{\Sigma_p})$. Indeed, one has

$$(41) \quad M_J(\underline{\lambda}_{\Sigma_p}) \cong M(\underline{\lambda}_{\Sigma_p \setminus J}) \otimes_E L(\underline{\lambda}_J).$$

By [28, Thm], one easily deduces from the above discussion:

Proposition B.4. *Keep the above notation and let π be a finite length smooth representation of T_p over E , then the locally \mathbb{Q}_p -analytic representation $\mathcal{F}_{\overline{B}_p}^{G_p}(M(\Delta_{\Sigma_p})^\vee, \pi)$ admits a decreasing filtration (where “ \vee ” denotes the dual in the BGG category $\mathcal{O}^{\overline{b}_{\Sigma_p}}$)*

$$\mathrm{Fil}^i \mathcal{F}_{\overline{B}_p}^{G_p}(M(\Delta_{\Sigma_p})^\vee, \pi) := \mathcal{F}_{\overline{B}_p}^{G_p}((\mathrm{Fil}^i M(\Delta_{\Sigma_p}))^\vee, \pi),$$

with $(\mathrm{Fil}^i / \mathrm{Fil}^{i+1}) \mathcal{F}_{\overline{B}_p}^{G_p}(M(\Delta_{\Sigma_p})^\vee, \pi) \cong \oplus_{J \subseteq N(\Delta), |J|=i} \mathcal{F}_{\overline{B}_p}^{G_p}(L(s_J \cdot \Delta_{\Sigma_p}), \pi)$.

Let ψ be a smooth character of T_p over E , we have in fact (cf. (24))

$$I(\psi \delta_{\Delta_{\Sigma_p}}) \cong \mathcal{F}_{\overline{B}_p}^{G_p}(L(-\Delta_{\Sigma_p}), \psi).$$

The following proposition follows from [11, Thm.4.3]:

Proposition B.5. *Keep the notation as above, let V be a very strongly admissible representation of G_p over E , π a finite length smooth representation of T_p over E , $J \subseteq N(\Delta) = N(-\Delta)$, then there exists a bijection*

$$\mathrm{Hom}_{G_p}(\mathcal{F}_{\overline{B}_p}^{G_p}(M_J(-\Delta_{\Sigma_p})^\vee, \pi), V(\Delta_J)) \xrightarrow{\sim} \mathrm{Hom}_{T_p}(\pi \otimes_E \delta_{B_p}, J_{B_p}(V(\Delta_J))).$$

Proof. By Breuil’s adjunction formula [11, Thm.4.3], one has

$$\mathrm{Hom}_{G_p}(\mathcal{F}_{\overline{B}_p}^{G_p}(M(-\Delta_{\Sigma_p})^\vee, \pi \otimes_E \delta_{B_p}^{-1}), V(\Delta_J)) \xrightarrow{\sim} \mathrm{Hom}_{T_p}(\pi \otimes_E \delta_{\Delta_{\Sigma_p}}, J_{B_p}(V(\Delta_J))),$$

however, since $V(\Delta_J)$ is $U(\mathfrak{g}_J)$ -finite, any map in the left set factors through the quotient (cf. (41))

$$\mathcal{F}_{\overline{B}_p}^{G_p}(M_J(-\Delta_{\Sigma_p})^\vee, \pi \otimes_E \delta_{B_p}^{-1}).$$

Indeed, by [28, Thm] and the discussion above Prop.B.4, any irreducible component of the kernel $\mathcal{F}_{\overline{B}_p}^{G_p}(M(-\Delta_{\Sigma_p})^\vee, \pi \otimes_E \delta_{B_p}^{-1}) \rightarrow \mathcal{F}_{\overline{B}_p}^{G_p}(M_J(-\Delta_{\Sigma_p})^\vee, \pi \otimes_E \delta_{B_p}^{-1})$ would be an irreducible component of a locally analytic representation of the form:

$$\mathcal{F}_{\overline{B}_p}^{G_p}(L(-s_{J'} \cdot \Delta_{\Sigma_p}), \psi) \cong I(\psi \delta_{s_{J'} \cdot \Delta_{\Sigma_p}})$$

where $J' \subseteq N(\Delta)$, $J' \cap J \neq \emptyset$, and ψ is a smooth character of T_p appearing as an irreducible component in $\pi \otimes_E \delta_{B_p}^{-1}$. While $I(\psi \delta_{s_{J'} \cdot \Delta_{\Sigma_p}})$ does not have non-zero $U(\mathfrak{g}_J)$ -finite vectors (e.g. by Schraen’s results [30, §2] and (25)), the proposition follows. \square

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